Prony, Ideals and Gauß Quadrature

Tomas Sauer

Lehrstuhl für Mathematik mit Schwerpunkt Digitale Bildverarbeitung
Fakultät für Informatik und Mathematik

FORWISS

Universität Passau

Fraunhofer IIS Department X-ray technology development EZRT

Joint work with Yuan Xu (University of Oregon)

Luminy, October 29, 2024

The Setup

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right)$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{l} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

The Setup

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right)$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{c} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{c} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

The Setup

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right)$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{l} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

The Setup

UNIVERSITÄT PASSAU

Moment sequences

1. Sequence $\mu = (\mu_\alpha : \alpha \in \mathbb{N}_0^s)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right) \Leftrightarrow$ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{l} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array} \right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array} \right)$$

The Setup

UNIVERSITÄT PASSAU

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right)$ ⇔ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{c} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{c} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

The Setup

UNIVERSITÄT PASSAU

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right) \Leftrightarrow$ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha x^\alpha \qquad \Rightarrow \qquad \ell(f) = \mu^T \hat{f} = \sum_{\alpha \in \mathbb{N}_0^s} \mu_\alpha \hat{f}_\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{c} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{c} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

The Setup

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s \right)$
2. Moments: $\mu_\alpha = \ell \left((\cdot)^\alpha \right) \Leftrightarrow$ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha \qquad \Rightarrow \qquad \ell(f) = \mu^T \hat{f} = \sum_{\alpha \in \mathbb{N}_0^s} \mu_\alpha \hat{f}_\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{c} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array} \right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{c} |\alpha| \le n \\ |\beta| \le n \end{array} \right)$$

The Setup

UNIVERSITÄT PASSAU

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right) \Leftrightarrow$ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha \, x^\alpha \qquad \Rightarrow \qquad \ell(f) = \mu^T \hat{f} = \sum_{\alpha \in \mathbb{N}_0^s} \mu_\alpha \hat{f}_\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{l} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

$$\ell(fg) = \hat{f}^T H \hat{g}, \qquad f, g \in \Pi$$

The Setup

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right) \Leftrightarrow$ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha x^\alpha \qquad \Rightarrow \qquad \ell(f) = \mu^T \hat{f} = \sum_{\alpha \in \mathbb{N}_0^s} \mu_\alpha \hat{f}_\alpha$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{l} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

$\ell(fg) = \hat{f}^T H_n \hat{g}, \quad f, g \in \Pi_n$

The Setup

Moment sequences

1. Sequence $\mu = \left(\mu_\alpha : \alpha \in \mathbb{N}_0^s\right)$
2. Moments: $\mu_\alpha = \ell\left((\cdot)^\alpha\right) \Leftrightarrow$ linear functional ℓ on $\Pi = \mathbb{C}[x]$
3. Polynomials:

$$f(x) = \sum_{\alpha \in \mathbb{N}_0^s} \hat{f}_\alpha x^\alpha \qquad \Rightarrow \qquad \ell(f) = \mu^T \hat{f} = \sum_{\alpha \in \mathbb{N}_0^s} \mu_\alpha \hat{f}_\alpha = e_0^T M_n \hat{f} = \ell(1 f)$$

Hankel operator/matrix

$$H = \left(\mu_{\alpha+\beta} : \begin{array}{l} \alpha \in \mathbb{N}_0^s \\ \beta \in \mathbb{N}_0^s \end{array}\right), \qquad H_n = \left(\mu_{\alpha+\beta} : \begin{array}{l} |\alpha| \le n \\ |\beta| \le n \end{array}\right)$$

$$\ell(fg) = \hat{f}^T H_n \hat{g}, \qquad f, g \in \Pi_n$$

Example: Quadrature

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha\, f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature/Cubature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature/Cubature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$
4. For $s > 1$ this works only in rare situations. Why?

Example: Quadrature

UNIVERSITÄT PASSAU

Approximation of functionals

1. Functional $\Pi \to \mathbb{R}$ **of finite rank**

$$\Theta(f) := \sum_{\alpha \in \mathbb{N}_0^s} \theta_\alpha \hat{f}_\alpha$$

2. **Exactness:** $\theta_\alpha = \mu_\alpha$, $\alpha \in \Gamma \subset \mathbb{N}_0^s$

Quadrature/Cubature

1. $\ell(f) = \int_\Omega f(x)\, w(x)\, dx$
2. $Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$
3. Exactness: $\Gamma = \Gamma_{2n} = \{\alpha : |\alpha| \le 2n\}$. $|\Gamma_{2n}| > 2|\Gamma_n|$
4. For $s > 1$ this works only in rare situations. Why? Numbers?

Gauß Quadrature

$$Q_n(f) = \sum_{|\alpha|\le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: x_α = zeros of orthogonal polynomial
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \frac{p(z)}{q(z)}$
3. Continued fractions . . .

Gauß Quadrature

$$Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: $x_\alpha =$ zeros of orthogonal polynomial
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \dfrac{p(z)}{q(z)}$
3. Continued fractions ...

Gauß Quadrature

$$Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: x_α = common zeros of orthogonal polynomials
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \dfrac{p(z)}{q(z)}$
3. Continued fractions . . .

Gauß Quadrature

$$Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: x_α = common zeros of orthogonal polynomials
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \frac{p(z)}{q(z)}$
3. Continued fractions . . .

Gauß Quadrature

$$Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: x_α = common zeros of orthogonal polynomials
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \frac{p(z)}{q(z)}$
3. Continued fractions ...

Gauß Quadrature

UNIVERSITÄT PASSAU

$$Q_n(f) = \sum_{|\alpha|\le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: x_α = common zeros of orthogonal polynomials
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \frac{p(z)}{q(z)}$
3. Continued fractions ...

Gauß Quadrature

UNIVERSITÄT PASSAU

$$Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

1. **Knots**: x_α = common zeros of orthogonal polynomials
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \dfrac{p(z)}{q(z)}$
3. Continued fractions ...

Gauß Quadrature

$$Q_n(f) = \sum_{|\alpha| \le n} w_\alpha f(\xi_\alpha)$$

Finding knots and weights

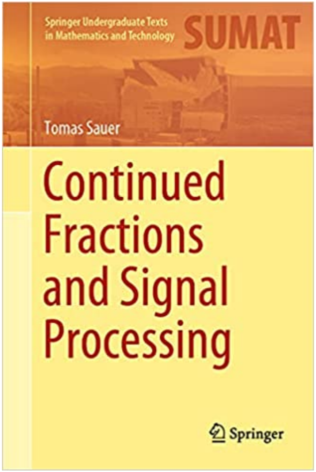
1. **Knots**: x_α = common zeros of orthogonal polynomials
2. **Weights**: interpolatory formula

Gauß' original approach

1. Moment generating function: $\mu(z) = \sum \mu_\alpha z^{-\alpha}$
2. Rational approximation $\mu(z) \approx \dfrac{p(z)}{q(z)}$
3. Continued fractions . . .

Advertisement

Want to know more?



A Very Special Case

Simplest integral

1. $\ell(f) = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1} f(x)\,dx$
2. Centrally symmetric (bad)
3. **No** common zeros of orthogonal polynomials.

A formula

$$\begin{aligned} Q(f) &= \frac{1}{36a+1} f(6a,6a) \\ &+ \frac{18a}{36a+1} f\left(-\frac{1}{36a}+\sqrt{\left(\frac{1}{36a}\right)^2+\frac{1}{6}}, -\frac{1}{36a}-\sqrt{\left(\frac{1}{36a}\right)^2+\frac{1}{6}}\right) \\ &+ \frac{18a}{36a+1} f\left(-\frac{1}{36a}-\sqrt{\left(\frac{1}{36a}\right)^2+\frac{1}{6}}, -\frac{1}{36a}+\sqrt{\left(\frac{1}{36a}\right)^2+\frac{1}{6}}\right) \end{aligned}$$

has quadratic accuracy. Why?

A Very Special Case

UNIVERSITÄT PASSAU

Simplest integral

1. $\ell(f) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x)\, dx$
2. Centrally symmetric (bad)
3. **No** common zeros of orthogonal polynomials.

A formula

$$
\begin{aligned}
Q(f) = {} & \frac{1}{36a+1} f(6a, 6a) \\
& + \frac{18a}{36a+1} f\left(-\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}} \right) \\
& + \frac{18a}{36a+1} f\left(-\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}} \right)
\end{aligned}
$$

has quadratic accuracy. Why?

A Very Special Case

Simplest integral

1. $\ell(f) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x)\, dx$
2. Centrally symmetric (bad)
3. **No** common zeros of orthogonal polynomials.

A formula

$$
\begin{aligned}
Q(f) = {} & \frac{1}{36a+1} f(6a, 6a) \\
& + \frac{18a}{36a+1} f\left(-\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}} \right) \\
& + \frac{18a}{36a+1} f\left(-\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}} \right)
\end{aligned}
$$

has quadratic accuracy. Why?

A Very Special Case

Simplest integral

1. $\ell(f) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x)\, dx$
2. Centrally symmetric (bad)
3. **No** common zeros of orthogonal polynomials.

A formula

$$
\begin{aligned}
Q(f) &= \frac{1}{36a+1} f(6a, 6a) \\
&+ \frac{18a}{36a+1} f\left(-\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}\right) \\
&+ \frac{18a}{36a+1} f\left(-\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}\right)
\end{aligned}
$$

has quadratic accuracy. Why?

A Very Special Case

Simplest integral

1. $\ell(f) = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} f(x)\, dx$
2. Centrally symmetric (bad)
3. **No** common zeros of orthogonal polynomials.

A formula

$$
\begin{aligned}
Q(f) = {} & \frac{1}{36a+1} f(6a, 6a) \\
& + \frac{18a}{36a+1} f\left(-\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}} \right) \\
& + \frac{18a}{36a+1} f\left(-\frac{1}{36a} - \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}}, -\frac{1}{36a} + \sqrt{\left(\frac{1}{36a}\right)^2 + \frac{1}{6}} \right)
\end{aligned}
$$

has quadratic accuracy. Why?

Flat Extension & Orthogonality

Canonical decomposition

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix}$$

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \rightarrow \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0, \quad \ell\left(P_n P_n^T\right)$ *nonsingular*
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \quad \to \quad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$, $\quad \ell\left(P_n P_n^T\right)$ *nonsingular,* $\quad \hat{P}_n = \ker H_n^\flat$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$, $\quad \ell\left(P_n P_n^T\right)$ *nonsingular*, $\quad \hat{P}_n = \ker H_n^\flat$
3. P_n **monic** orthogonal basis for inner product

Flat Extension & Orthogonality

UNIVERSITÄT PASSAU

Canonical decomposition & flat extension

$$H_n = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n} \end{pmatrix} \qquad \to \qquad H_n^\flat = \begin{pmatrix} H_{n-1} & H_{n,n-1} \\ H_{n,n-1}^T & H_{n,n-1}^T H_{n-1}^{-1} H_{n,n-1} \end{pmatrix}$$

Schur complement, **but:** is $H_n^\flat$ Hankel?

Orthogonality

1. *Vector polynomials* P_n with coefficients $\hat{P}_n := \begin{pmatrix} -H_{n-1}^{-1} H_{n,n-1} \\ I \end{pmatrix}$
2. $\ell\left(\Pi_{n-1} P_n^T\right) = 0$, $\quad \ell\left(P_n P_n^T\right)$ *nonsingular*, $\quad \hat{P}_n = \ker H_n^\flat$
3. P_n **monic** orthogonal basis for **signed** inner product

Multiplication and Recurrence

UNIVERSITÄT PASSAU

Multplication

$$x_j P_n(x) = A_{n,j} P_{n+1}(x) + B_{n,j} P_n(x) + C_{n,j} P_{n-1}(x) + \cdots + E_{n,j} P_0(x)$$

1. **Three term** recurrence due to orthogonality
2. $A_{n,j}$ and $C_{n,j}$ have **maximal rank**

Consistency

Commuting of multiplication $x_k \left(x_j P_n(x) \right) = x_j \left(x_k P_n(x) \right)$ implies

$$\begin{aligned}
A_{n-1,j} A_{n,k} &= A_{n-1,k} A_{n,j} \\
A_{n-1,j} B_{n,k} + B_{n-1,j} A_{n-1,k} &= A_{n-1,k} B_{n,j} + B_{n-1,k} A_{n-1,j} \\
A_{n-1,j} C_{n,k} + B_{n-1,j} B_{n-1,k} + C_{n-1,j} A_{n-2,k} &= A_{n-1,k} C_{n,j} + B_{n-1,k} B_{n-1,j} + C_{n-1,k} A_{n-2,j} \\
B_{n-1,j} C_{n-1,k} + C_{n-1,j} B_{n-2,k} &= B_{n-1,k} C_{n-1,j} + C_{n-1,k} B_{n-2,j} \\
C_{n-1,j} C_{n-2,k} &= C_{n-1,k} C_{n-2,j}.
\end{aligned}$$

Multiplication and Recurrence

Multplication

$$x_j P_n(x) = A_{n,j} P_{n+1}(x) + B_{n,j} P_n(x) + C_{n,j} P_{n-1}(x)$$

1. **Three term** recurrence due to orthogonality
2. $A_{n,j}$ and $C_{n,j}$ have **maximal rank**

Consistency

Commuting of multiplication $x_k \left(x_j P_n(x) \right) = x_j \left(x_k P_n(x) \right)$ implies

$$\begin{aligned}
A_{n-1,j} A_{n,k} &= A_{n-1,k} A_{n,j} \\
A_{n-1,j} B_{n,k} + B_{n-1,j} A_{n-1,k} &= A_{n-1,k} B_{n,j} + B_{n-1,k} A_{n-1,j} \\
A_{n-1,j} C_{n,k} + B_{n-1,j} B_{n-1,k} + C_{n-1,j} A_{n-2,k} &= A_{n-1,k} C_{n,j} + B_{n-1,k} B_{n-1,j} + C_{n-1,k} A_{n-2,j} \\
B_{n-1,j} C_{n-1,k} + C_{n-1,j} B_{n-2,k} &= B_{n-1,k} C_{n-1,j} + C_{n-1,k} B_{n-2,j} \\
C_{n-1,j} C_{n-2,k} &= C_{n-1,k} C_{n-2,j}.
\end{aligned}$$

Multiplication and Recurrence

UNIVERSITÄT PASSAU

Multplication

$$x_j P_n(x) = A_{n,j}P_{n+1}(x) + B_{n,j}P_n(x) + C_{n,j}P_{n-1}(x)$$

1. **Three term** recurrence due to orthogonality
2. $A_{n,j}$ and $C_{n,j}$ have **maximal rank**

Consistency

Commuting of multiplication $x_k\left(x_j P_n(x)\right) = x_j\left(x_k P_n(x)\right)$ implies

$$\begin{aligned}
A_{n-1,j}A_{n,k} &= A_{n-1,k}A_{n,j} \\
A_{n-1,j}B_{n,k} + B_{n-1,j}A_{n-1,k} &= A_{n-1,k}B_{n,j} + B_{n-1,k}A_{n-1,j} \\
A_{n-1,j}C_{n,k} + B_{n-1,j}B_{n-1,k} + C_{n-1,j}A_{n-2,k} &= A_{n-1,k}C_{n,j} + B_{n-1,k}B_{n-1,j} + C_{n-1,k}A_{n-2,j} \\
B_{n-1,j}C_{n-1,k} + C_{n-1,j}B_{n-2,k} &= B_{n-1,k}C_{n-1,j} + C_{n-1,k}B_{n-2,j} \\
C_{n-1,j}C_{n-2,k} &= C_{n-1,k}C_{n-2,j}.
\end{aligned}$$

Multiplication and Recurrence

UNIVERSITÄT PASSAU

Multplication

$$x_j P_n(x) = A_{n,j} P_{n+1}(x) + B_{n,j} P_n(x) + C_{n,j} P_{n-1}(x)$$

1. **Three term** recurrence due to orthogonality
2. $A_{n,j}$ and $C_{n,j}$ have **maximal rank**

Consistency

Commuting of multiplication $x_k \left(x_j P_n(x) \right) = x_j \left(x_k P_n(x) \right)$ implies

$$\begin{aligned}
A_{n-1,j} A_{n,k} &= A_{n-1,k} A_{n,j} \\
A_{n-1,j} B_{n,k} + B_{n-1,j} A_{n-1,k} &= A_{n-1,k} B_{n,j} + B_{n-1,k} A_{n-1,j} \\
A_{n-1,j} C_{n,k} + B_{n-1,j} B_{n-1,k} + C_{n-1,j} A_{n-2,k} &= A_{n-1,k} C_{n,j} + B_{n-1,k} B_{n-1,j} + C_{n-1,k} A_{n-2,j} \\
B_{n-1,j} C_{n-1,k} + C_{n-1,j} B_{n-2,k} &= B_{n-1,k} C_{n-1,j} + C_{n-1,k} B_{n-2,j} \\
C_{n-1,j} C_{n-2,k} &= C_{n-1,k} C_{n-2,j}.
\end{aligned}$$

Time for a Summary

UNIVERSITÄT PASSAU

Observation

1. If μ is definite then there exist **monic** orthogonal polynomials P_n with respect to μ and their recurrence coefficients satisfy

$$\begin{aligned} A_{n-1,j}A_{n,k} &= A_{n-1,k}A_{n,j} \\ A_{n-1,j}B_{n,k}+B_{n-1,j}A_{n-1,k} &= A_{n-1,k}B_{n,j}+B_{n-1,k}A_{n-1,j} \\ A_{n-1,j}C_{n,k}+B_{n-1,j}B_{n-1,k}+C_{n-1,j}A_{n-2,k} &= A_{n-1,k}C_{n,j}+B_{n-1,k}B_{n-1,j}+C_{n-1,k}A_{n-2,j} \\ B_{n-1,j}C_{n-1,k}+C_{n-1,j}B_{n-2,k} &= B_{n-1,k}C_{n-1,j}+C_{n-1,k}B_{n-2,j} \\ C_{n-1,j}C_{n-2,k} &= C_{n-1,k}C_{n-2,j}. \end{aligned} \qquad (\heartsuit)$$

Necessary for consistency!

2. Moreover: Commuting of multiplication in $\Pi / \langle P_n \rangle$
3. Important **additional** condition is the commuting

$$A_{n-1,j}C_{n,k} = A_{n-1,k}C_{n,j} \qquad (\heartsuit^+)$$

4. **None** of this conditions occurs in $s = 1$

Time for a Summary

UNIVERSITÄT PASSAU

Observation

1. If μ is definite then there exist **monic** orthogonal polynomials P_n with respect to μ and their recurrence coefficients satisfy

$$\begin{aligned}
A_{n-1,j}A_{n,k} &= A_{n-1,k}A_{n,j} \\
A_{n-1,j}B_{n,k}+B_{n-1,j}A_{n-1,k} &= A_{n-1,k}B_{n,j}+B_{n-1,k}A_{n-1,j} \\
A_{n-1,j}C_{n,k}+B_{n-1,j}B_{n-1,k}+C_{n-1,j}A_{n-2,k} &= A_{n-1,k}C_{n,j}+B_{n-1,k}B_{n-1,j}+C_{n-1,k}A_{n-2,j} \\
B_{n-1,j}C_{n-1,k}+C_{n-1,j}B_{n-2,k} &= B_{n-1,k}C_{n-1,j}+C_{n-1,k}B_{n-2,j} \\
C_{n-1,j}C_{n-2,k} &= C_{n-1,k}C_{n-2,j}.
\end{aligned} \qquad (\heartsuit)$$

Necessary for consistency!

2. Moreover: Commuting of multiplication in $\Pi / \langle P_n \rangle$
3. Important **additional** condition is the commuting

$$A_{n-1,j}C_{n,k} = A_{n-1,k}C_{n,j} \qquad (\heartsuit^+)$$

4. **None** of this conditions occurs in $s=1$

Time for a Summary

UNIVERSITÄT PASSAU

Observation

1. If μ is definite then there exist **monic** orthogonal polynomials P_n with respect to μ and their recurrence coefficients satisfy

$$\begin{aligned} A_{n-1,j}A_{n,k} &= A_{n-1,k}A_{n,j} \\ A_{n-1,j}B_{n,k}+B_{n-1,j}A_{n-1,k} &= A_{n-1,k}B_{n,j}+B_{n-1,k}A_{n-1,j} \\ A_{n-1,j}C_{n,k}+B_{n-1,j}B_{n-1,k}+C_{n-1,j}A_{n-2,k} &= A_{n-1,k}C_{n,j}+B_{n-1,k}B_{n-1,j}+C_{n-1,k}A_{n-2,j} \\ B_{n-1,j}C_{n-1,k}+C_{n-1,j}B_{n-2,k} &= B_{n-1,k}C_{n-1,j}+C_{n-1,k}B_{n-2,j} \\ C_{n-1,j}C_{n-2,k} &= C_{n-1,k}C_{n-2,j}. \end{aligned} \qquad (\heartsuit)$$

Necessary for consistency!

2. Moreover: Commuting of multiplication in $\Pi / \langle P_n \rangle$
3. Important **additional** condition is the commuting

$$A_{n-1,j}C_{n,k} = A_{n-1,k}C_{n,j} \qquad (\heartsuit^+)$$

4. **None** of this conditions occurs in $s = 1$

Time for a Summary

UNIVERSITÄT PASSAU

Observation

1. If μ is definite then there exist **monic** orthogonal polynomials P_n with respect to μ and their recurrence coefficients satisfy

$$\begin{aligned} A_{n-1,j}A_{n,k} &= A_{n-1,k}A_{n,j} \\ A_{n-1,j}B_{n,k}+B_{n-1,j}A_{n-1,k} &= A_{n-1,k}B_{n,j}+B_{n-1,k}A_{n-1,j} \\ A_{n-1,j}C_{n,k}+B_{n-1,j}B_{n-1,k}+C_{n-1,j}A_{n-2,k} &= A_{n-1,k}C_{n,j}+B_{n-1,k}B_{n-1,j}+C_{n-1,k}A_{n-2,j} \\ B_{n-1,j}C_{n-1,k}+C_{n-1,j}B_{n-2,k} &= B_{n-1,k}C_{n-1,j}+C_{n-1,k}B_{n-2,j} \\ C_{n-1,j}C_{n-2,k} &= C_{n-1,k}C_{n-2,j}. \end{aligned} \qquad (\heartsuit)$$

Necessary for consistency!

2. Moreover: Commuting of multiplication in $\Pi / \langle P_n \rangle$
3. Important **additional** condition is the commuting

$$A_{n-1,j}C_{n,k} = A_{n-1,k}C_{n,j} \qquad (\heartsuit^+)$$

4. **None** of this conditions occurs in $s = 1$

Time for a Summary

UNIVERSITÄT PASSAU

Observation

1. If μ is definite then there exist **monic** orthogonal polynomials P_n with respect to μ and their recurrence coefficients satisfy

$$\begin{aligned} A_{n-1,j}A_{n,k} &= A_{n-1,k}A_{n,j} \\ A_{n-1,j}B_{n,k}+B_{n-1,j}A_{n-1,k} &= A_{n-1,k}B_{n,j}+B_{n-1,k}A_{n-1,j} \\ A_{n-1,j}C_{n,k}+B_{n-1,j}B_{n-1,k}+C_{n-1,j}A_{n-2,k} &= A_{n-1,k}C_{n,j}+B_{n-1,k}B_{n-1,j}+C_{n-1,k}A_{n-2,j} \\ B_{n-1,j}C_{n-1,k}+C_{n-1,j}B_{n-2,k} &= B_{n-1,k}C_{n-1,j}+C_{n-1,k}B_{n-2,j} \\ C_{n-1,j}C_{n-2,k} &= C_{n-1,k}C_{n-2,j}. \end{aligned} \qquad (\heartsuit)$$

Necessary for consistency!

2. Moreover: Commuting of multiplication in $\Pi / \langle P_n \rangle$
3. Important **additional** condition is the commuting

$$A_{n-1,j}C_{n,k} = A_{n-1,k}C_{n,j} \qquad (\heartsuit^+)$$

4. **None** of this conditions occurs in $s = 1$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff ($\heartsuit^+$) holds true.

The reason . . .

1. Characterization of “good bases” by commuting
2. Multiplication tables

$$M_j := \left(I \quad - H_{n-1}^{-1} H_{n,n-1} \right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f,\ f \in \Pi_{n-1}$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff ($\heartsuit^+$) holds true.

The reason ...

1. Characterization of "good bases" by commuting
2. Multiplication tables

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1}\right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f,\, f \in \Pi_{n-1}$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff ($\heartsuit^+$) holds true.

The reason . . .

1. Characterization of "good bases" by commuting
2. Multiplication tables

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1}\right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f,\ f \in \Pi_{n-1}$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff ($\heartsuit^+$) holds true.

The reason . . .

1. Characterization of “good bases” by commuting
2. Multiplication tables

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1}\right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f,\ f \in \Pi_{n-1}$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff (♡$^+$) holds true.

The reason . . .

1. Characterization of “good bases” by commuting
2. Multiplication tables/Companion matrices

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1} \right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f$, $f \in \Pi_{n-1}$

A Bit of Algebra

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff (♡$^+$) holds true.

The reason . . .

1. Characterization of "good bases" by commuting
2. Multiplication tables/Companion matrices

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1} \right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f,\ f \in \Pi_{n-1}$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff ($\heartsuit^+$) holds true.

The reason ...

1. Characterization of "good bases" by commuting
2. Multiplication tables/Companion matrices

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1}\right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f, f \in \Pi_{n-1}$

A Bit of Algebra

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff (♡$^+$) holds true.

The reason . . .

1. Characterization of "good bases" by commuting
2. Multiplication tables/Companion matrices

$$M_j := \left(I \quad -H_{n-1}^{-1} H_{n,n-1} \right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute $\quad \Leftrightarrow \quad \Pi_{n-1} = \Pi / \langle P_n \rangle$
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f,\ f \in \Pi_{n-1}$

A Bit of Algebra

UNIVERSITÄT PASSAU

Theorem

Suppose recurrence matrices satisfy (♡). Then P_n, $n \in \mathbb{N}_0$, are H-bases iff ($\heartsuit^+$) holds true.

The reason . . .

1. Characterization of "good bases" by commuting
2. Multiplication tables/Companion matrices

$$M_j := \left(I \quad - H_{n-1}^{-1} H_{n,n-1} \right) L_{n-1,j}, \qquad L_{n,j} = \sum_{|\alpha|=n} e_{\alpha+\epsilon_j} e_\alpha$$

3. P_n H-basis iff M_j commute $\Leftrightarrow$ common zeros
4. Based on $f(M) := \sum_\alpha \hat{f}_\alpha M_1^{\alpha_1} \cdots M_s^{\alpha_1} \quad \Rightarrow \quad f(M)1 = f, f \in \Pi_{n-1}$

The Way Back

UNIVERSITÄT PASSAU

Ideals and zeros

Suppose that (♡) + (♡$^+$) hold:

1. $\Pi / \langle P_{n+1} \rangle = \Pi_n$
2. P_{n+1} has $r_n = \dim \Pi_n$ common zeros:

$$(\zeta, \mathscr{Q}_\zeta), \qquad \sum_\zeta \dim \mathscr{Q}_\zeta = r_n$$

3. Recall: $\mathscr{Q}_\zeta$ is D-invariant space, **multiplicity**

Consequence

Nonsingular **Vandermonde matrix**

$$V_n := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ |\alpha| \le n \end{array} \right)$$

The Way Back

UNIVERSITÄT PASSAU

Ideals and zeros

Suppose that (♡) + (♡$^+$) hold:

1. $\Pi / \langle P_{n+1} \rangle = \Pi_n$
2. P_{n+1} has $r_n = \dim \Pi_n$ common zeros:

$$(\zeta, \mathscr{Q}_\zeta), \qquad \sum_\zeta \dim \mathscr{Q}_\zeta = r_n$$

3. Recall: $\mathscr{Q}_\zeta$ is D-invariant space, **multiplicity**

Consequence

Nonsingular **Vandermonde matrix**

$$V_n := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ |\alpha| \le n \end{array} \right)$$

The Way Back

Ideals and zeros

Suppose that (♡) + ($\heartsuit^+$) hold:

1. $\Pi / \langle P_{n+1} \rangle = \Pi_n$
2. P_{n+1} has $r_n = \dim \Pi_n$ common zeros:

$$(\zeta, \mathscr{Q}_\zeta), \qquad \sum_\zeta \dim \mathscr{Q}_\zeta = r_n$$

3. Recall: $\mathscr{Q}_\zeta$ is D-invariant space, **multiplicity**

Consequence

Nonsingular **Vandermonde matrix**

$$V_n := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ |\alpha| \le n \end{array} \right)$$

The Way Back

Ideals and zeros

Suppose that (♡) + (♡$^+$) hold:

1. $\Pi / \langle P_{n+1} \rangle = \Pi_n$
2. P_{n+1} has $r_n = \dim \Pi_n$ common zeros:

$$(\zeta, \mathscr{Q}_\zeta), \qquad \sum_\zeta \dim \mathscr{Q}_\zeta = r_n$$

3. Recall: $\mathscr{Q}_\zeta$ is D-invariant space, **multiplicity**

Consequence

Nonsingular **Vandermonde matrix**

$$V_n := \left(\left(q(D) (\cdot)^\alpha \right) (\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ |\alpha| \le n \end{array} \right)$$

The Way Back

UNIVERSITÄT PASSAU

Ideals and zeros

Suppose that (♡) + ($♡^+$) hold:

1. $\Pi / \langle P_{n+1} \rangle = \Pi_n$
2. P_{n+1} has $r_n = \dim \Pi_n$ common zeros:

$$(\zeta, \mathscr{Q}_\zeta), \qquad \sum_\zeta \dim \mathscr{Q}_\zeta = r_n$$

3. Recall: $\mathscr{Q}_\zeta$ is D-invariant space, **multiplicity**. **Simple** ζ: $\mathscr{Q}_\zeta = \{1\}$

Consequence

Nonsingular **Vandermonde matrix**

$$V_n := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ |\alpha| \le n \end{array} \right)$$

The Way Back

UNIVERSITÄT PASSAU

Ideals and zeros

Suppose that (♡) + (♡$^+$) hold:

1. $\Pi / \langle P_{n+1} \rangle = \Pi_n$
2. P_{n+1} has $r_n = \dim \Pi_n$ common zeros:

$$(\zeta, \mathscr{Q}_\zeta), \qquad \sum_\zeta \dim \mathscr{Q}_\zeta = r_n$$

3. Recall: $\mathscr{Q}_\zeta$ is D-invariant space, **multiplicity**. **Simple** ζ: $\mathscr{Q}_\zeta = \{1\}$

Consequence

Nonsingular **Vandermonde matrix**

$$V_n := \left(\left(q(D)\, (\cdot)^\alpha \right) (\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ |\alpha| \le n \end{array} \right)$$

Back to Prony

Recovery issue

$$f(x) = \sum_{\zeta \in Z} f_\zeta(x)\, \zeta^x, \qquad f_\zeta \in \mathcal{Q}_\zeta$$

from $\mu_\alpha := f(\alpha)$, $\alpha \in \mathbb{N}_0^s$.

Prony ideal

1. Prony ideal: $H\hat{f} = 0$, $f \in \mathscr{I}$
2. Factorization:

$$H = V_\infty^T D V_\infty, \qquad V_\infty := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathcal{Q}_\zeta, \zeta \\ \alpha \in \mathbb{N}_0^s \end{array} \right)$$

with D block diagonal of rank $\dim \Pi / \langle \mathscr{I} \rangle$

Back to Prony

UNIVERSITÄT PASSAU

Recovery issue

$$f(x) = \sum_{\zeta \in Z} f_\zeta(x)\, \zeta^x, \qquad f_\zeta \in \mathscr{Q}_\zeta$$

from $\mu_\alpha := f(\alpha)$, $\alpha \in \mathbb{N}_0^s$.

Prony ideal

1. Prony ideal: $H\hat{f} = 0$, $f \in \mathscr{I}$
2. Factorization:

$$H = V_\infty^T D V_\infty, \qquad V_\infty := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ \alpha \in \mathbb{N}_0^s \end{array} \right)$$

with D block diagonal of rank $\dim \Pi / \langle \mathscr{I} \rangle$

Back to Prony

UNIVERSITÄT PASSAU

Recovery issue

$$f(x) = \sum_{\zeta \in Z} f_\zeta(x)\, \zeta^x, \qquad f_\zeta \in \mathscr{Q}_\zeta$$

from $\mu_\alpha := f(\alpha)$, $\alpha \in \mathbb{N}_0^s$.

Prony ideal

1. Prony ideal: $H\hat{f} = 0$, $f \in \mathscr{I}$
2. Factorization:

$$H = V_\infty^T D V_\infty, \qquad V_\infty := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ \alpha \in \mathbb{N}_0^s \end{array} \right)$$

with D block diagonal of rank $\dim \Pi / \langle \mathscr{I} \rangle$

Back to Prony

Recovery issue

$$f(x) = \sum_{\zeta \in Z} f_\zeta(x)\, \zeta^x, \qquad f_\zeta \in \mathscr{Q}_\zeta$$

from $\mu_\alpha := f(\alpha)$, $\alpha \in \mathbb{N}_0^s$.

Prony ideal

1. Prony ideal: $H\hat{f} = 0$, $f \in \mathscr{I}$
2. Factorization:

$$H = V_\infty^T D V_\infty, \qquad V_\infty := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{matrix} q \in \mathscr{Q}_\zeta, \zeta \\ \alpha \in \mathbb{N}_0^s \end{matrix} \right)$$

with D block diagonal of rank $\dim \Pi / \langle \mathscr{I} \rangle$

Back to Prony

Recovery issue

$$f(x) = \sum_{\zeta \in Z} f_\zeta(x)\,\zeta^x, \qquad f_\zeta \in \mathscr{Q}_\zeta$$

from $\mu_\alpha := f(\alpha)$, $\alpha \in \mathbb{N}_0^s$.

Prony ideal

1. Prony ideal: $H\hat{f} = 0$, $f \in \mathscr{I}$
2. Factorization:

$$H = V_\infty^T D V_\infty, \qquad V_\infty := \left(\left(q(D)(\cdot)^\alpha \right)(\zeta) : \begin{array}{c} q \in \mathscr{Q}_\zeta, \zeta \\ \alpha \in \mathbb{N}_0^s \end{array} \right)$$

with D block diagonal of rank $\dim \Pi / \langle \mathscr{I} \rangle$ → (Curto & Fialkow)

Putting Things Together

What we got so far

Assume (♡) + (♡$^+$) and rank conditions

1. The P_{n+1} are H-bases
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\,\zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \quad |\alpha| \le n$$

Putting Things Together

What we got so far

Assume (♡) + (♡$^+$) and rank conditions

1. The P_{n+1} are H-bases
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \qquad |\alpha| \le n$$

Putting Things Together

What we got so far

Assume (♡) + (♡$^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\,\zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \qquad |\alpha| \le n$$

Putting Things Together

What we got so far

Assume (♡) + ($\heartsuit^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \qquad |\alpha| \le n$$

Putting Things Together

What we got so far

Assume (♡) + ($♡^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \qquad |\alpha| \le n$$

Putting Things Together

What we got so far

Assume (♡) + ($\heartsuit^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \quad |\alpha| \le n$$

Putting Things Together

What we got so far

Assume (♡) + ($♡^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \quad |\alpha| \le n \qquad \Leftrightarrow \qquad \left(\mu_\alpha^n : |\alpha| \le n\right) = V_n^T \hat{f}_{n+1,\zeta}$$

Putting Things Together

What we got so far

Assume (♡) + ($♡^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \quad |\alpha| \le n \qquad \Leftrightarrow \qquad \left(\mu_\alpha^n : |\alpha| \le n\right) = V_n^T \hat{f}_{n+1,\zeta}$$

→ sequence μ^n of moment sequences

Putting Things Together

What we got so far

Assume (♡) + ($♡^+$) and rank conditions

1. The P_{n+1} are H-bases and have r_n common zeros Z_{n+1}
2. Use them as exponents ζ for Prony function

$$f_{n+1}(x) = \sum_{\zeta \in Z_{n+1}} f_{n+1,\zeta}(x)\, \zeta^x$$

3. Sample: $\mu_\alpha^{n+1} = f_{n+1}(\alpha)$

Connecting the levels

$$f_{n+1}(\alpha) = \mu_\alpha^n, \quad |\alpha| \le n \qquad \Leftrightarrow \qquad \left(\mu_\alpha^n : |\alpha| \le n\right) = V_n^T \hat{f}_{n+1,\zeta}$$

→ sequence μ^n of moment sequences, $\mu_\alpha^n = \mu_\alpha^{n+1}$, $|\alpha| \le n$

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + (♡$^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$

$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$

2. $H_n^\flat$ is Hankel

Theorem Equivalences

1. μ is definite and (♡$^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + (♡$^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
2. $H_n^\flat$ is Hankel

Theorem Equivalences

1. μ is definite and (♡$^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

Theorem

Assume (♡) + ($♡^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$

$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$

Gauss cubature

2. $H_n^\flat$ is Hankel

Theorem Equivalences

1. μ is definite and ($♡^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($♡^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel

Theorem Equivalences

1. μ is definite and ($♡^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

Theorem

Assume (♡) + ($♡^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu^n_\alpha = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H^\flat_n$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($♡^+$) holds
2. μ is definite and $H^\flat_n$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($\heartsuit^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($\heartsuit^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($♡^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($♡^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($♡^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($♡^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($\heartsuit^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($\heartsuit^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($\heartsuit^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($\heartsuit^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

More Is True

UNIVERSITÄT PASSAU

Theorem

Assume (♡) + ($♡^+$) and rank conditions. Then

1. for $\mu = \lim \mu^n$
$$\mu_\alpha^n = \mu_\alpha, \qquad |\alpha| \le 2n-1$$
Gauss cubature
2. $H_n^\flat$ is Hankel and defines cubature moments

Theorem Equivalences

1. μ is definite and ($♡^+$) holds
2. μ is definite and $H_n^\flat$ is Hankel
3. μ is definite and each H_n admits a flat extension
4. μ is definite and the M_j commute
5. There exists Gauss cubature

And the Gauß Quadrature?

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$
5. Works for $s = 2$ and $n = 1, 2$

And the Gauß Quadrature?

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\begin{pmatrix} I & M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \end{pmatrix} L_{n-1,j}$
5. Works for $s = 2$ and $n = 1, 2$

And the Gauß Quadrature?

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$
5. Works for $s = 2$ and $n = 1, 2$

And the Gauß Quadrature?

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\begin{pmatrix} I & M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \end{pmatrix} L_{n-1,j}$
5. Works for $s = 2$ and $n = 1, 2$

And the Gauß Quadrature?

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix} =: \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$
5. Works for $s=2$ and $n=1,2$

And the Gauß Quadrature?

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix} =: \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$
5. Works for $s = 2$ and $n = 1, 2$

And the Gauß Quadrature?

UNIVERSITÄT PASSAU

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix} =: \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$: quadratic equations
5. Works for $s=2$ and $n=1,2$

And the Gauß Quadrature?

UNIVERSITÄT PASSAU

Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix} =: \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$: quadratic equations
5. Works for $s = 2$ and $n = 1, 2$

And the Gauß Quadrature?

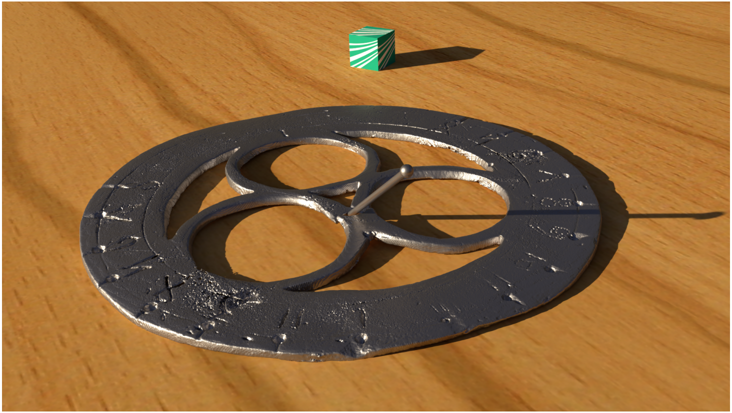
Simple observation

1. H_{n-1} depends on μ_α, $|\alpha| \le 2n-2$.
2. $H_{n,n-1}$ can be decomposed

$$H_{n,n-1} = \begin{pmatrix} \mu_{\alpha+\beta} : |\alpha| = 0, |\beta| = n-1 \\ \vdots \\ \mu_{\alpha+\beta} : |\alpha| = n-1, |\beta| = n-1 \\ \mu_{\alpha+\beta} : |\alpha| = n, |\beta| = n-1 \end{pmatrix} =: \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix}$$

3. Free parameter for extension of H_{n-1}
4. Commute $\left(I \quad M_{n-1}^{-1} \begin{pmatrix} H_{<n,n-1} \\ \mu_{2n-1}^T \end{pmatrix} \right) L_{n-1,j}$: quadratic equations
5. Works for $s = 2$ and $n = 1,2$. Challenge for CAS

Time's up



\end – **questions?**