Optimal spline spaces and outlier-removal strategies in isogeometric analysis

Espen Sande[1]

[1]Institute of Mathematics, EPFL

- M. S. Floater and E. S., *Optimal spline spaces for L^2 n-width problems with boundary conditions,* Constr. Approx. (2019).
- E. S., C. Manni and H. Speleers, *Sharp error estimates for spline approximation: explicit constants, n-widths, and eigenfunction convergence,* Math. Models Methods Appl. Sci. (2019).
- A. Bressan and E. S., *Approximation in FEM, DG and IGA: A Theoretical Comparison,* Numer. Math. (2019).
- Y. Voet, E. S. and A. Buffa, *A mathematical theory for mass lumping and its generalization with applications to isogeometric analysis*, Comput. Methods Appl. Mech. Engrg. (2023).
- Y. Voet, E. S. and A. Buffa, *Robust mass lumping and outlier removal strategies in isogeometric analysis*, (preprint).

Motivating problem

We look for $u\colon \Omega \times [0, T] \to \mathbb{R}$ such that

$$
\begin{aligned}
\partial^2_{tt} u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) &= f(\mathbf{x}, t) && \text{in } \Omega \times (0, T], \\
u(\mathbf{x}, t) &= 0 && \text{on } \partial\Omega \times (0, T], \\
u(\mathbf{x}, 0) &= u_0(\mathbf{x}) && \text{in } \Omega, \\
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Looking for a discrete approximation $u_n(\mathbf{x}, t) = \sum_{i=1}^{n} u_i(t) B_i(\mathbf{x})$ and using a Galerkin method leads to the semi-discrete problem

$$
\begin{aligned}
M\ddot{\mathbf{u}}(t) + K\mathbf{u}(t) &= \mathbf{f}(t) && \text{for } t \in [0, T], \\
\mathbf{u}(0) &= \mathbf{u}_0, \\
\dot{\mathbf{u}}(0) &= \mathbf{v}_0,
\end{aligned}
$$

where

$$
M_{ij} = \int_\Omega B_i(\mathbf{x}) B_j(\mathbf{x}) d\mathbf{x}, \quad K_{ij} = \int_\Omega \nabla B_i(\mathbf{x}) \cdot \nabla B_j(\mathbf{x}) d\mathbf{x}
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Challenges:

- This involves computing M^{-1} which can be computationally very costly.
- The largest stable time-step depends inversely on the largest eigenvalue $\lambda_n(K, M)$ which can become quite large.

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Requirement: Smallest eigenvalues must not change!

Overview

Minimizing $\lambda_n(K, M)$

Mass lumping in Isogeometric Analysis

Minimizing $\lambda_n(K, \tilde{M})$

Spline spaces

Knot vector Ξ:

$$0 = \xi_0 < \xi_1 < \cdots < \xi_N < \xi_{N+1} = 1,$$

$$I_j := [\xi_j, \xi_{j+1}),\ j = 0, \ldots, N-2,\ I_N := [\xi_{N-1}, 1]$$

$$h := \max_{0 \le j \le N-1} (\xi_{j+1} - \xi_j)$$

Spline space of degree p and smoothness k:

$$\mathbb{S}^k_{p,\Xi} := \{ s \in C^k[0,1] : s|_{I_j} \in \mathbb{P}_p,\ j = 0, 1, \ldots, N-1 \}$$

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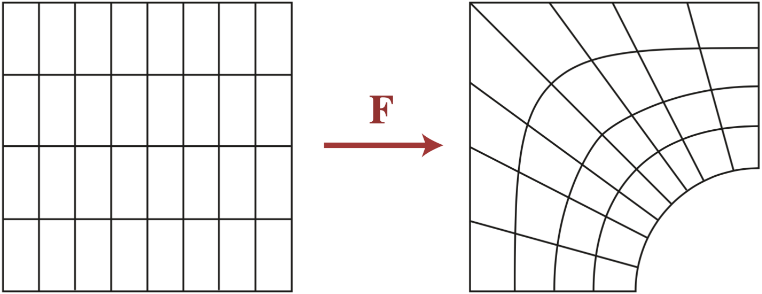
Norm:

$\|\cdot\|$: L^2-norm

Isogeometric Analysis



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- In general the domain is divided into several patches, each described by a geometric mapping F_i.

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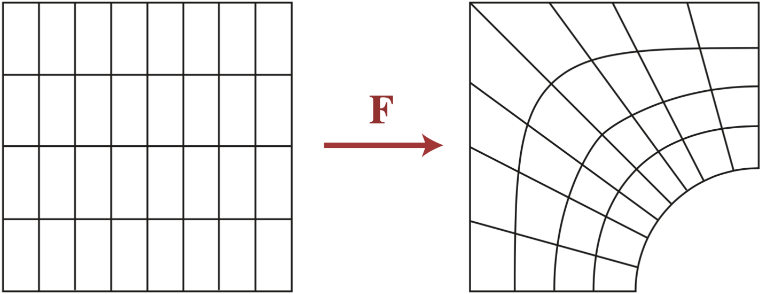

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Due to the (local) tensor-product structure, we will first consider 1D.

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$$\lambda_n(K, M) \approx \lambda_n(-\Delta).$$

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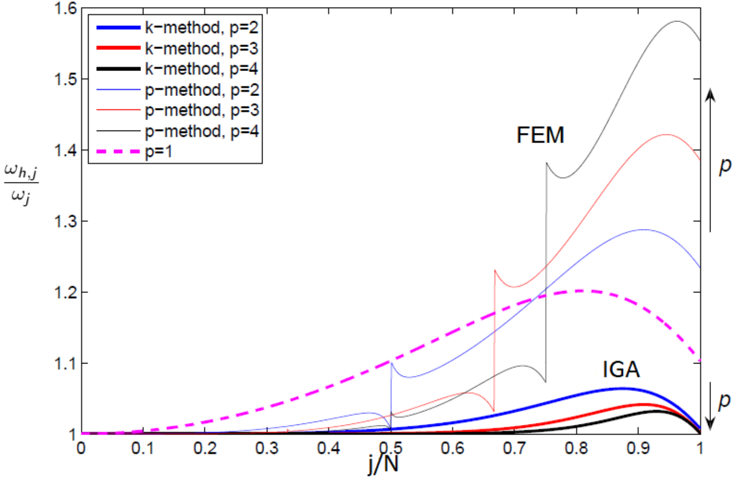
Define

- $\omega_j := \lambda_j(-\Delta)^{1/2}$
- $\omega_{h,j} := \lambda_j(K, M)^{1/2}$



- $p - k$ branches
- only a single branch approximates the true spectrum
- maximal smoothness ($k = p - 1$) no spurious branches

See [Cottrell et al. 2006], [Hughes et al. 2008], [Garoni et al. 2019]...

Outliers

....however there is a problem for large j.

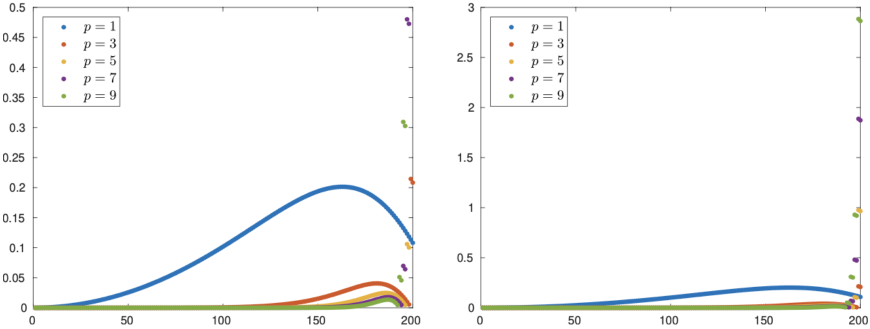


Figure: Relative error ($\frac{\omega_{h,j} - \omega_j}{\omega_j}$) in the case $k = p - 1$ and $n = 200$.

See [Cottrell et al. 2006], [Hughes et al. 2008], [Hughes et al. 2014], [Gallistl et al. 2017] [Chan and Evans 2018] etc.

Kolmogorov n-widths

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- For a class of functions $A \subset L^2$ and an n-dimensional subspace $\mathbb{X}_n \subset L^2$, let

$$E(A, \mathbb{X}_n) = \sup_{u \in A} \|u - \Pi_n u\|,$$

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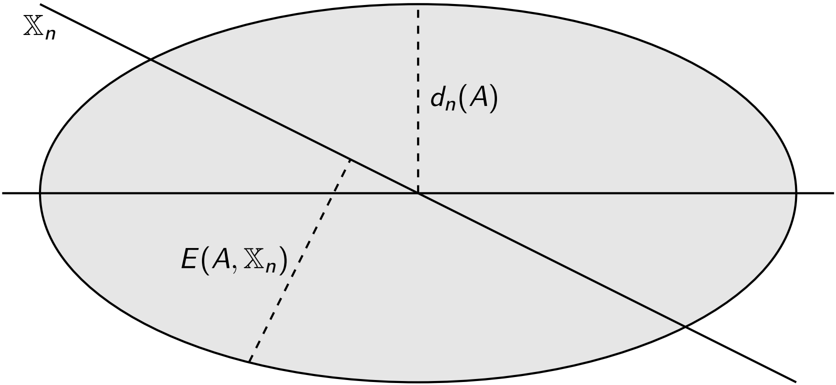
- The subspace $\mathbb{X}_n$ is called an optimal subspace for A if

$$d_n(A) = E(A, \mathbb{X}_n).$$

Kolmogorov n-widths



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Note that the constant $C = d_n(A^r)$ can also be achieved for other projections than Π_n (for example for Ritz projections).

Optimal spaces

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$$\frac{E(A^r, \mathbb{S}_{p,\boldsymbol{\tau}})}{d_n(A^r)} \leq \frac{\left(\frac{1}{(n-p)\pi}\right)^r}{\left(\frac{1}{n\pi}\right)^r} = \left(\frac{1}{1-\frac{p}{n}}\right)^r \xrightarrow[n\to\infty]{} 1, \quad \forall p \geq r-1.$$

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- The above limit is not 1 for C^0 (FEM/SEM) or C^{-1} (DG) [Bressan and S. 2019].

Example: Other boundary conditions

Define

$$A_0^r := \{u \in H^r : \|u^{(r)}\| \le 1,\ u^{(\alpha)}(0) = u^{(\alpha)}(1) = 0,\ 0 \le \alpha < r,\ \alpha \text{ even}\}$$
$$A_1^r := \{u \in H^r : \|u^{(r)}\| \le 1,\ u^{(\alpha)}(0) = u^{(\alpha)}(1) = 0,\ 0 \le \alpha < r,\ \alpha \text{ odd}\}.$$

Then

$$d_n(A_0^r) = \frac{1}{(n+1)^r \pi^r}, \qquad d_n(A_1^r) = \frac{1}{(n\pi)^r},$$

and the classical optimal spaces are

$$\mathbb{T}_{n,0} = \mathrm{span}\{\sin(\pi x), \sin(2\pi x), \ldots, \sin(n\pi x)\},$$
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The optimal spline spaces are [Floater and S. 2019]

$$\mathbb{S}_{p,n,0} := \{s \in \mathbb{S}_{p,\boldsymbol{\tau}_0} : s^{(\alpha)}(0) = s^{(\alpha)}(1) = 0, \quad 0 \leq \alpha \leq p, \quad \alpha \text{ even}\},$$
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Even derivatives

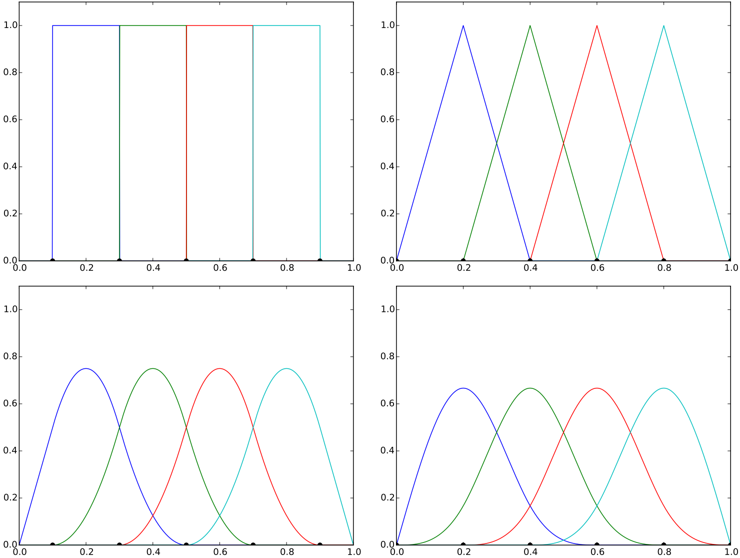

Figure: Basis functions for $\mathbb{S}_{p,n,0}$ of degree $p = 0, 1, 2, 3$ for $n = 4$.

Odd derivatives

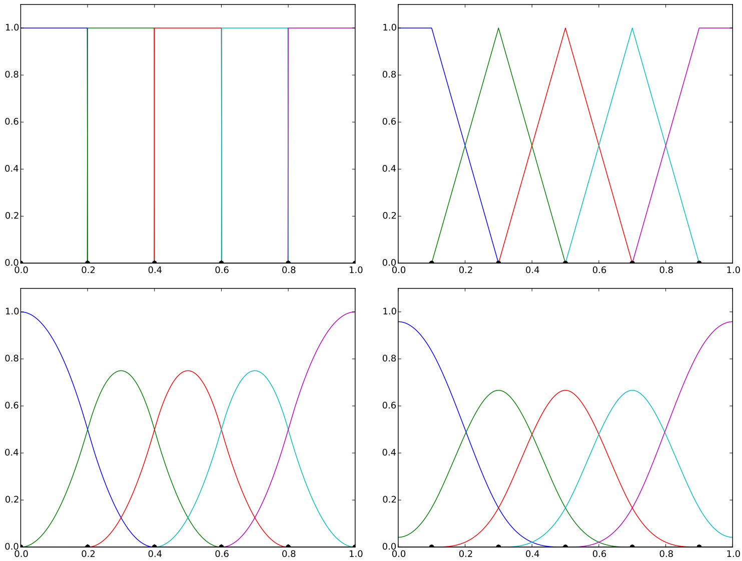

Figure: Basis functions for $\mathbb{S}_{p,n,1}$ of degree $p = 0, 1, 2, 3$ for $n = 5$. Similar to the 'reduced spline spaces' in [Takacs and Takacs 2016]

Optimal spline spaces solve the outlier problem

Solve $K\mathbf{u}_j = \omega_{h,j}^2 M\mathbf{u}_j$ in $\mathbb{S}_{p,n,0}$, then we have shown that

$$\omega_j \leq \omega_{h,j} \leq \frac{\omega_j}{1 - \left(\frac{\omega_j}{\omega_{n+1}}\right)^{p+1}},$$

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Main ingredients of the proof:

- $\mathbb{S}_{p,n,0}$ is optimal for A_0^r.
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The argument also covers the tensor-product case and the biharmonic/polyharmonic case [Manni, S., Speleers 2023].

Eigenvalue problem in higher dimensions

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Theorem (Voet, S. and Buffa)

For all $0 \leq k \leq p-1$ *and any* $j = 1, \dots, n$ *we have*

$$\frac{\|u_j - P^{\mathbf{k}}_{\mathbf{p}} u_j\|_{L^2}}{\|u_j\|_{L^2}} \leq C_{PDE} C_{Geo} \frac{C^{Spline}_{p,k,1}}{n} \omega_j.$$

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Theorem (Voet, S. and Buffa)

For all $0 \leq k \leq p-1$ *and any* $j = 1, \dots, n$ *such that* u_j *is smooth we have*

$$\frac{\|u_j - P^{\mathbf{k}}_{\mathbf{p}} u_j\|_{L^2}}{\|u_j\|_{L^2}} \leq C_{PDE} C_{Geo} C^{Spline}_{p,k,p+1} \left(\frac{1}{n}\right)^{p+1} \omega_j^{p+1},$$

Spline approximation constant

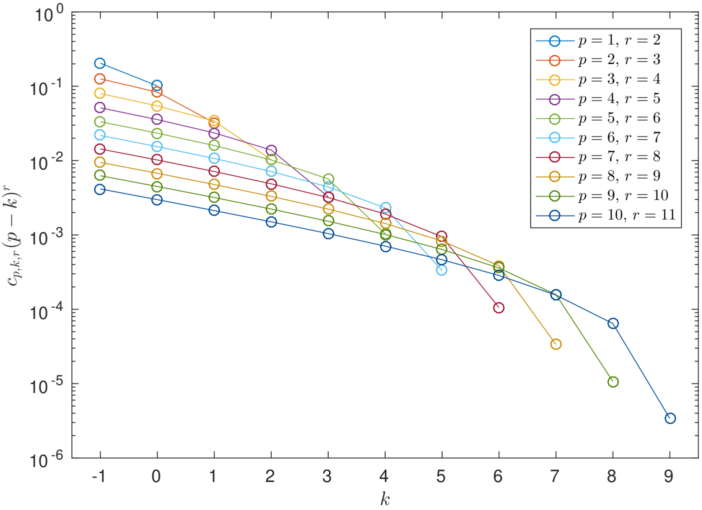

Figure: Numerical values of $C^{\text{Spline}}_{p,k,r}$ for $r = p + 1$ and different choices of $p \geq 1$ and $-1 \leq k \leq p - 1$, see [S., Manni, Speleers 2020].

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- In IGA, using B-splines with together with their dual basis is proposed in [Anitescu et al. 2019], however it has many drawbacks.
- In [Voet, S. and Buffa 2023] we use different block-wise lumping strategies to obtain several lumped mass matrices.

Mass lumping in 1D

Let

$$M = D_i + R_i$$

- D_i: all super and sub-diagonals with distance from the diagonal smaller than i.
- R_i: remainder.

Our lumped mass matrices:

$$P_i = D_i + \mathcal{L}(R_i), \quad i = 1, \ldots, n$$

- $\mathcal{L}$: diagonal row-sum.
- $P_1 = \mathcal{L}(M)$.

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- R_i: remainder.

Our lumped mass matrices:

$$P_i = D_i + \mathcal{L}(R_i), \quad i = 1, \ldots, n$$

- $\mathcal{L}$: diagonal row-sum.
- $P_1 = \mathcal{L}(M)$.

We have shown that

$$\lambda_k(K, P_1) \leq \cdots \leq \lambda_k(K, P_n) = \lambda_k(K, M), \quad \forall k = 1, 2, \ldots, n. \tag{2}$$

Example: Laplacian in 1D

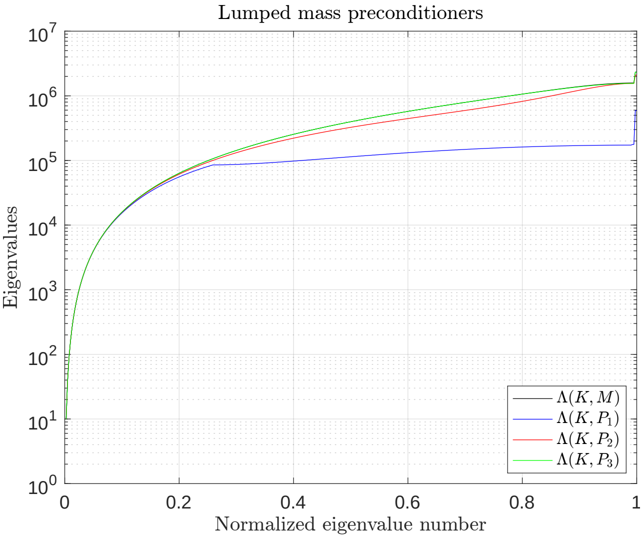

Figure: Comparison of $\Lambda(K, M)$ and $\Lambda(K, P_i)$ for $i = 1, 2, 3$ and $p = 3$

Mass lumping in higher dimensions: Strategy 1

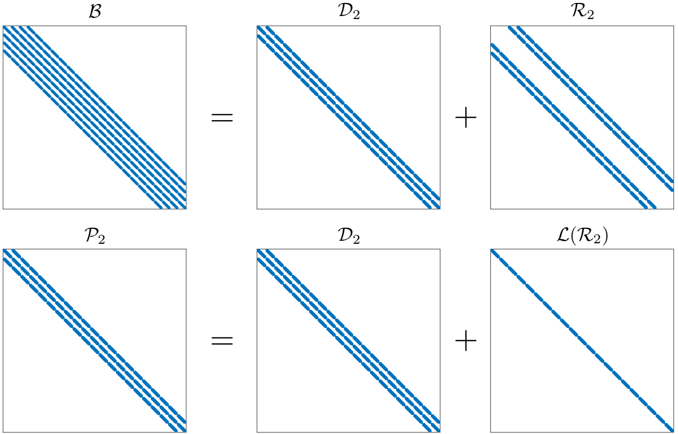

Figure: Example: Block tridiagonal matrix $\mathcal{P}_2$ constructed from a block septadiagonal matrix $\mathcal{B}$

$\mathcal{P}_i$: Lumps all super and sub blocks with block-distance $\geq i$ from the block-diagonal.

Mass scaling

We propose a classical deflation technique going back to [Hotelling 1943].

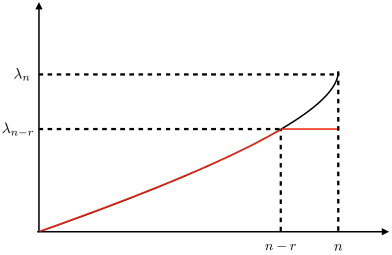

Figure: Truncation of the largest eigenvalues

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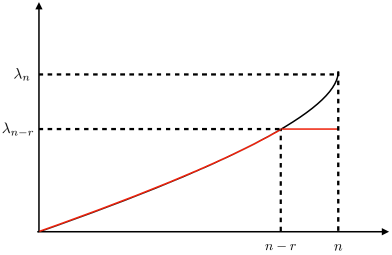


Figure: Truncation of the largest eigenvalues

It requires computing the last r eigenvalues $\lambda_j(K, P_i)$, $j = n - r + 1, \ldots n$ and their corresponding eigenfunctions. Using Lanczos method has essentially the same computational cost as time-stepping forward.

Example: Laplacian on trimmed domain

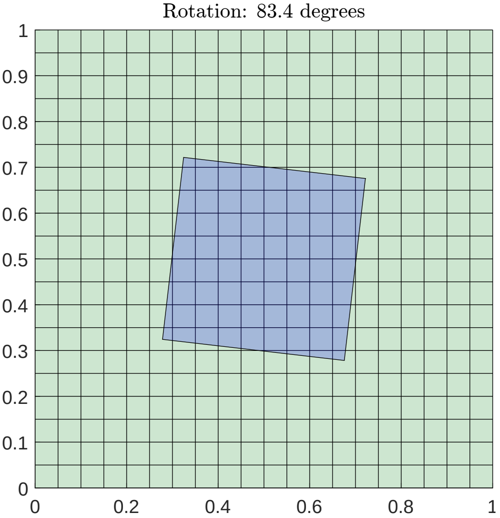
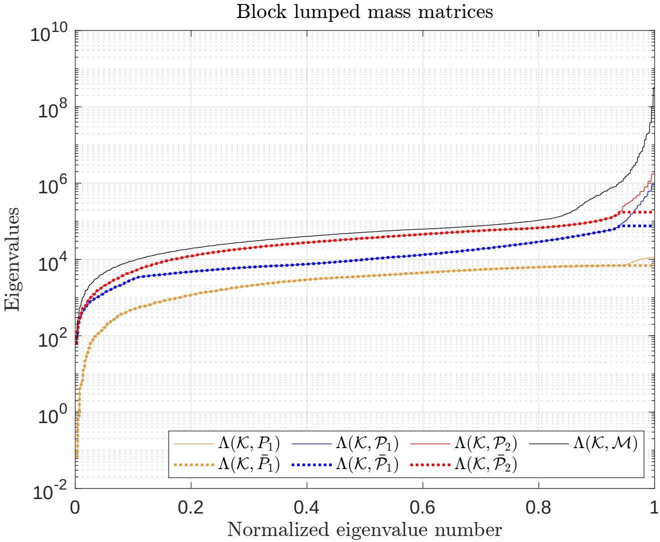
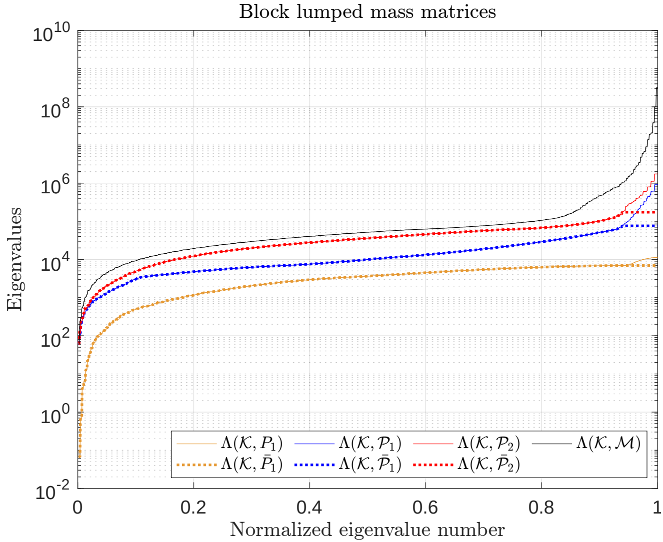


Figure: Shifted and rotated square

Example: Laplacian on trimmed domain for $p = 3$



Example: Laplacian on trimmed domain for $p = 3$



It was observed in [Leidinger 2020] that the diagonal row-sum lumping strategy does not need to be mass scaled when trimming. However, there are outliers among the lowest eigenvalues.

Conclusion

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Conclusion

- We have related the outlier problem in Isogeometric Analysis to the n-width problem and explained why smooth splines provide a good approximation of a large part of the spectrum of a differential operator.
- We have generalized the classical diagonal row-sum mass lumping strategy for use in Isogeometric Analysis.
- Due to the relatively small amount of outlier-frequencies when using maximally smooth splines we have seen that a classical deflation technique works very well in this case.

Thank you for your attention!