#### Hierarchical Matrices for 3D Helmholtz problems in multi-patch IGA-BEM setting

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### **Outline**

- Boundary element methods and integral formulation (BIE)
- Isogeometric setting (IGA-BEM) and multi-patch geometries
- **•** Hierarchical matrix formulation
- $\bullet$  Low rank approximation of admissible blocks
- Some numerical results
- $\bullet$  Current developments.



### Boundary Element Methods

- Boundary Element Methods are numerical methods to solve PDEs and can be seen in some cases as a valid alternative to classical domain methods as Finite Element or Finite Difference methods.
- The differential problem is reformulated into Boundary Integral Equations which require suitable and efficient *quadrature formulae* for their solution

#### **Advantages:**

- Reduced dimension of the computational domain: easier computation on complex geometries (no domain mesh generation!)
- Simplicity to solve external problems: easier treatment of unbounded domains

#### **•** Disadvantages:

- Fundamental solution of the PDE problem is needed beforehand ٠
- Singular kernels (singular integrals) ۰
- Fully populated matrices ۰



## 3D Acoustic model problem

We consider 3D acoustic problems described by the Helmholtz equation, with Neumann boundary conditions:

**D Acoustic model problem**  
\nWe consider 3D acoustic problems described by the Helmholtz equation, with  
\ndeumann boundary conditions:  
\n
$$
\begin{cases}\n\Delta u(x) + \kappa^2 u(x) = f(x) & \text{in } \Omega \subset \mathbb{R}^3 \\
\frac{\partial u(x)}{\partial n_x} = u_N(x) & \text{on } \Gamma\n\end{cases}
$$
\n
$$
= u_N(x)
$$
\n
$$
= u_N(x)
$$
\nFor exterior problems, the acoustic domain  $\Omega$  is infinite. The unknown function u at infinity must satisfy the Sommerfeld radiation condition:  
\n
$$
\lim_{|x| \to \infty} |x| \left( \nabla u(x) \cdot \frac{x}{|x|} - i\kappa u(x) \right) = 0
$$
\n
$$
= 0
$$
\n $$ 

Acoustic parameters:  $\omega$  angular frequency, c speed of the wave Frequency–domain:  $\kappa = \omega/c$ , wave number,  $\lambda = 2\pi/k$ , wave length



• For exterior problems, the acoustic domain  $\Omega$  is infinite.  $\implies$  the unknown function u at infinity must satisfy the Sommerfeld radiation condition:

$$
\lim_{|\mathbf{x}| \to \infty} |\mathbf{x}| \left( \nabla u(\mathbf{x}) \cdot \frac{\mathbf{x}}{|\mathbf{x}|} - i \kappa u(\mathbf{x}) \right) = 0
$$

any radiated or scattered acoustic wave has to converge towards zero when the radius tends to infinity.



### Boundary integral equation

● Setting null the external body forces (f=0), we consider a direct integral representation formula for u.

**boundary integral equation**

\nsetting null the external body forces (f=0), we consider a direct integral presentation formula for u.

\n
$$
u(\mathbf{x}) = \int_{\Gamma} \frac{\partial}{\partial n_y} \mathcal{G}_k(\mathbf{x}, \mathbf{y}) u(\mathbf{y}) d\gamma_y - \int_{\Gamma} \mathcal{G}_k(\mathbf{x}, \mathbf{y}) \frac{\partial u}{\partial n} (\mathbf{y}) d\gamma_y \quad \mathbf{x} \in \Omega \setminus \Gamma
$$
\nis a single layer potential

\nto the integral representation formula is strictly connected to the definition of the

\nodamental solution  $\mathcal{G}$  and its normal derivative. Setting  $r = ||\mathbf{x} - \mathbf{y}||$ 

\n
$$
v_{\mathbf{x}}(\mathbf{x}, \mathbf{y}) := \frac{e^{i\kappa r}}{4\pi r}, \qquad \frac{\partial}{\partial n_y} \mathcal{G}_k(\mathbf{x}, \mathbf{y}) = \frac{e^{i\kappa r}}{4\pi r} \left( -\frac{1}{r} + i\kappa \right) \frac{\partial r}{\partial n_y}, \qquad \frac{\partial r}{\partial n_y} = -\frac{r \cdot n_y}{r}
$$
\nobplying the trace operator we get the Conventional Boundary Integral Equation

\n
$$
\int_{\Gamma} \mathcal{G}_k(\mathbf{x}, \mathbf{y}) u_N(\mathbf{y}) d\gamma_y = \frac{1}{2} \varphi(\mathbf{x}) + \int_{\Gamma} \frac{\partial}{\partial n_y} \mathcal{G}_k(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d\gamma_y \qquad \mathbf{x} \in \Gamma, \varphi := u_{|_{\Gamma}}
$$
\nthe radiation condition is included in the integral formulation

\n**Solution**

\n**3.1**

\n**6.1**

\n**6.1**

\n**6.1**

\n**6.1**

\n**6.1**

\n**7.1**

\n**8.2**

\n**9.1**

\n**1.1**

\n**1.1**

\n**1.1**

\n**2.1**

\n**2.1**

\n**3.2**

\n<

The integral representation formula is strictly connected to the definition of the fundamental solution G and its normal derivative. Setting  $r = ||x - y||$ 

$$
\mathcal{G}_{\kappa}(\mathbf{x}, \mathbf{y}) \coloneqq \frac{e^{i\kappa r}}{4\pi r}, \qquad \frac{\partial}{\partial \mathbf{n}_{\gamma}} \mathcal{G}_{\kappa}(\mathbf{x}, \mathbf{y}) = \frac{e^{i\kappa r}}{4\pi r} \left( -\frac{1}{r} + i\kappa \right) \frac{\partial r}{\partial \mathbf{n}_{\gamma}}, \qquad \frac{\partial r}{\partial \mathbf{n}_{\gamma}} = -\frac{r \cdot \mathbf{n}_{\gamma}}{r}
$$

• applying the trace operator we get the Conventional Boundary Integral Equation CBIE: al<br>  $\sqrt{\frac{\Gamma}{r}}$ <br>
on of the<br>  $\frac{\Gamma \cdot \mathbf{n}_y}{r}$ <br>
al Equation<br>  $\in \Gamma, \varphi \coloneqq U_{|_{\Gamma}}$ 

$$
\int_{\Gamma} \mathcal{G}_{\kappa}(\mathbf{x}, \mathbf{y}) u_{N}(\mathbf{y}) d\gamma_{\gamma} = \frac{1}{2} \varphi(\mathbf{x}) + \int_{\Gamma} \frac{\partial}{\partial \mathbf{n}_{\gamma}} \mathcal{G}_{\kappa}(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d\gamma_{\gamma}
$$

$$
\pmb{X}\in\varGamma,\,\varphi\coloneqq\pmb{U}_{\big|_{\varGamma}}
$$

The radiation condition is included in the integral formulation



### Isogeometric analysis

• Standard approach: CAD geometry is replaced by FEM geometry (mesh)

• The mesh is an approximate geometry: many problems (thin shell structures, boundary layer in fluids) are very sensitive to geometric imperfections



#### • Isoparametric approach:

*the solution space for dependent variables is represented in terms of the same functions which represent the geometry*" **[Cottrell, Hughes, Bazilevs; CMAME 2005]**

The goal was to develop an analysis framework based on functions capable of exactly representing geometry

• IDEA: exploit CAD techniques and representations





• Isogeometric Analysis has led a new interest also in possible applications in the BEM context (previously only considered in FEM)

• The possibility of describing accurately both the geometry and the solution has been studied also in the BEM approach (IGA-BEM)

**[Politis, Ginnis, Kaklis et al, 2009], [ Simpson, Scott, et al., 2012, 2014]**

- The use of IgA in the BEM context can radically improve the corresponding numerical schemes because of the additional smoothness of NURBS and Bsplines in comparison to  $C^0$ -continuous piecewise polynomials
- Representation of 3D objects only needs to be encompassed by their *boundary surfaces* based on Boundary representation (B-rep).
- To approximate accurately the integrals coming from the IGA-BEM formulation we have constructed new appropriate quadrature schemes, tailored on Bsplines.



**[Aimi, Calabrò, Falini, S., Sestini,** CMAME **2020] [Falini, Giannelli, Kanduc, S., Sestini,** Int.J. Num.Met. Eng. **2019]**

#### IgA multi-patch boundary representation

The boundary  $\Gamma$  is a union of M patches  $\Gamma = \prod_{i=1}^m \Gamma^{(i)}$ ,  $\ell \neq k$   $\Rightarrow \frac{1}{2} \partial \Gamma^{(i)} \cap \partial \Gamma^{(k)} = \Omega$  common edge, **corner point or 1** and 1 and empty set and the corner point or set and the set of th<br>Experimental set of the set of th  $\begin{aligned} \mathbf{R} &\mathbf{tion} \\ &\xrightarrow{(l)} \bigcap_{\mathcal{O}} \varGamma^{(k)} = \varnothing \\ &\xrightarrow{\Gamma^{(l)}} \bigcap \partial \varGamma^{(k)} = \varnothing \end{aligned}$ **presentation**<br>
(*i*),  $\ell \neq k$   $\Rightarrow$   $\begin{cases} \n\frac{\sum_{i=1}^{n} (f^{(i)}) \bigcap_{i=1}^{n} f^{(i)}}{\sum_{i=1}^{n} (f^{(i)}) \bigcap_{i=1}^{n} f^{(i)}} = \text{common edge,} \\ \n\text{for } p \text{ is the empty set}\n\end{cases}$ <br>
S representation **y representation**<br>  $\Gamma = \bigcup_{\ell=1}^{M} \Gamma^{(\ell)}, \ell \neq k \Rightarrow \begin{cases} \frac{\sum_{\ell}^{(\ell)} \bigcap \Gamma^{(k)} = \varnothing}{\varnothing \Gamma^{(\ell)} \bigcap \varnothing \Gamma^{(k)} = \text{ common edge,} \\ \text{ geometric mapping} \end{cases}$ <br>
NURBS representation<br>
asis of bi-degree **d**<sub>c</sub> (clamped knot vectors) Tepresentation<br>
=  $\bigcup_{\ell=1}^{M} \Gamma^{(\ell)}$ ,  $\ell \neq k$   $\Rightarrow$   $\left\{ \begin{aligned} &\frac{\sum_{\ell=1}^{(\ell)} \bigcap_{\ell} \Gamma^{(k)} = \varnothing}{\partial \Gamma^{(\ell)} \bigcap \partial \Gamma^{(k)} = \varnothing} \text{ common edge,} \\ &\text{geometric mapping} \end{aligned} \right.$ <br>
NURBS representation<br>
s of bi-degree **d**<sub>*s*</sub> (clamped knot vectors)  $\overline{a}$  $k)$   $\alpha$ *M*  $\sqrt{a} \sqrt{b} \sqrt{b} \sqrt{b} \sqrt{b} \sqrt{b}$  $k \Rightarrow \{^{0}I \}$  $\begin{bmatrix} 0,1 \end{bmatrix}^{\epsilon} \rightarrow \varGamma^{(\ell)}$  ge **11: -patch boundary representation**<br>
dary  $\Gamma$  is a union of *M* patches  $\Gamma = \bigcup_{i=1}^{M} \Gamma^{(i)}$ ,  $\ell \neq k$   $\Rightarrow$   $\begin{cases} \Gamma^{(i)} \cap \Gamma^{(k)} = \varnothing \\ \partial \Gamma^{(i)} \cap \partial \Gamma^{(k)} = \varnothing \\ \end{cases}$ <br>  $\overline{\Gamma}^{(i)} = \text{Image}(\mathbf{F}^{(i)}, \mathbf{F}^{(i)} : [0,1]^2 \rightarrow \overline{\Gamma}^{$ 

**(***i***) - patch boundary representing the patch of** *M* **patches**  $\lim_{t \to 1} \Gamma^{(i)}$ **<br>
(***i***) =** *Image* **(<b>F**<sup>(*i*</sup>), F<sup>(*i*</sup>) :  $[0,1]^2 \rightarrow \overline{F}^{(i)}$  geometr<br>  $\frac{\sum_{i \in \mathcal{I}_0^{(i)}} w_i^{(i)} \mathbf{Q}_i^{(i)} \mathbf{\hat{B}}_{i,d_g}^{(i)}(\mathbf{t})}{\sum_{i \in \mathcal$  $^{2}$ , NURBS representation  $\overline{a}$  $\sum_{i} W_i^{(\ell)} \hat{B}_{i,d_g}^{(\ell)}(t)$  ,  $\zeta \in [0,1]$ , *g*  $\Gamma$  is a<br>  $\Gamma$  is a<br>  $\Gamma$  is a<br>  $\Gamma$ <br>  $\Gamma$  is a<br>  $\Gamma$ <br>  $\Gamma$  $dg$   $\left( \begin{array}{c} 0 \\ 1 \end{array} \right)$  $\begin{aligned} \mathbf{Q} & \mathbf{Q} \mathbf{Q}^{(i)} \\ \mathbf{Q} & \mathbf{Q}^{(i)} \\ \mathbf{Q}^{(i)} & \mathbf{Q}^{(i)} \\ \mathbf{Q}^{(i)} & \mathbf{Q}^{(i)} \\ \mathbf{Q}^{(i)} & \mathbf{Q}^{(i)} \\ \mathbf{Q} & \mathbf{Q}^{(i)}$ **a** union of *M*<br> *ge* (F<sup>(c)</sup>), F<sup>(c)</sup>:<br>  $\frac{(\ell) \mathbf{Q_i^{(\ell)}} \hat{B}_{i,d_g}^{(\ell)}(\mathbf{t})}{W_i^{(\ell)} \hat{B}_{i,d_g}^{(\ell)}(\mathbf{t})}$ ,<br>  $=$  tensor production **)**<br> **)**<br> **)**<br> **)**<br> **)**<br> **(i)**<br> **(d)**<br> **atch boul**<br> **a** union of *M* pa<br>
<u>ige</u>(F<sup>(i)</sup>), F<sup>(i)</sup> : [0,1]<br>  $\frac{(\ell)\mathbf{Q}^{(\ell)}_i\hat{B}^{(\ell)}_{i,d_g}(t)}{W^{(\ell)}_i\hat{B}^{(\ell)}_{i,d_g}(t)}, \quad t \in \mathbb{R}$ <br>
= tensor product B  $i \in \mathcal{I}_{\alpha}^{(\ell)}$  $i \in \mathcal{I}^{(\ell)}_{\sigma}$ **t**

 $\left\{\mathsf{B}^{(\ell)}_{\mathsf{i},\mathsf{d}_g},\, \mathsf{i}\in\mathcal{I}_g^{(\ell)}\right\}=1$  tensor product B-spline basis of bi-degree  $\mathsf{d}_g$  (clamped knot vectors)



**Discretization**  
\n
$$
S_{\mathbf{d},h} := \text{span} \{ B_{\mathbf{j},\mathbf{d}}^{(\ell)} : j \in \mathcal{J}^{(\ell)}, 1 \leq \ell \leq M \} \text{ IgA spline discretization space}
$$
\n
$$
B_{\mathbf{j},\mathbf{d}}^{(\ell)}(\mathbf{x}) = \hat{B}_{\mathbf{j},\mathbf{d}}^{(\ell)} \circ \mathbf{F}^{(\ell)^{-1}}(\mathbf{x}) \qquad \mathbf{x} \in \Gamma^{(\ell)} \text{ lifted bases}
$$
\n• Free knot vector selection on each patch  $\Rightarrow$  inter-patch adaptivity\n
$$
N_{DOF} = \sum_{\ell=1}^{M} |J^{(\ell)}| \qquad \varphi := U_{\mathbf{l}_F}
$$

 $\bullet$  Free knot vector selection on each patch  $\Rightarrow$  inter-patch adaptivity

**Discretization**  
\n
$$
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\n
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$$
\n
$$
\bullet \text{ Free knot vector selection on each patch} \Rightarrow \text{inter-patch adaptivity}
$$
\n
$$
N_{DOF} = \sum_{\ell=1}^{M} \left| \mathcal{J}^{(\ell)} \right| \qquad \varphi := U_{\mathbf{j},\mathbf{r}}
$$
\n
$$
\varphi_h \approx \varphi, \quad \varphi_h(\mathbf{x}) := \sum_{\ell=1}^{M} \sum_{j \in \mathcal{J}^{(\ell)}} \alpha_j^{(\ell)} B_{\mathbf{j},\mathbf{d}}^{(\ell)}(\mathbf{x}), \qquad \mathbf{x} \in \Gamma
$$
\nPlipsing

\nHamiltonian integral to the same term, i.e.,  $\mathbf{F} \in \mathcal{F}$  is a function of the graph. The set of  $\mathbf{F}$  is the same term, i.e.,  $\mathbf{F} \in \mathcal{F}$ , and  $\mathbf{F} \in \mathcal{F}$  is a function of  $\mathbf{F}$  and  $\mathbf{F} \in \mathcal{F}$ .



### The linear system

• Domain is parametrized with M patches:

**net**<br>  $\vec{r}$ <br>  $\vec{r}$   $\vec{r$  $\begin{aligned} \n\textbf{N}\textbf{C}\textbf{S}\textbf{S}\text{ parametric} \n\overset{(e)}{=} [\mathbf{0},1]^2 \rightarrow \n\overset{(e)}{=} \mathbf{F}^{(\ell)}(\textbf{s}), \n\overset{(e)}{=} \mathbf{F}^{(\ell)}(\textbf{t}), \n\end{aligned}$  $\begin{aligned} \n\textbf{N}\textbf{C}\textbf{S}\textbf{S}\text{ parametric} \n\overset{(e)}{=} [\textbf{0},\textbf{1}]^2 \rightarrow \n\overset{(e)}{=} \textbf{F}^{(e)}(\textbf{s}), \n\overset{(e)}{=} \textbf{F}^{(e)}(\textbf{t}), \n\overset{(e)}{=} \left|\frac{\partial \textbf{F}^{(e)}}{\partial t_1}\right| \n\end{aligned}$ Syster<br>
trized with M<br>  $\rightarrow \mathbb{R}^3$ <br>  $\rightarrow$  s = (s<sub>1</sub>,s<sub>2</sub>)<br>  $\rightarrow$  t = (t<sub>1</sub>,t<sub>2</sub>)<br>  $\rightarrow$ <br>  $\rightarrow$   $\frac{\partial F^{(\ell)}}{\partial t_2}$ <br>  $\rightarrow$  F<sup>( $\ell$ )</sup>-1)(x)  $\mathbf{R}^{(l)}$  is para<br>  $\mathbf{R}^{(l)} = \mathbf{R}^{(l)}$ <br>  $\mathbf{R}^{(l)} = \mathbf{F}$ <br>  $\mathbf{R}^{(l)}(\mathbf{t}) :=$ <br>  $\sum_{j \in \mathcal{J}} \alpha_j^{(l)}$ <br>  $\mathbf{R}^{(l)}$  ix entrop  $\boldsymbol{F}^{(\ell)} = [0,1]^2 \to \mathbb{R}^3$ **r system**<br>
metrized with M patches<br>  $\begin{aligned}\n\mathbf{F} \rightarrow \mathbb{R}^3 \\
(\mathbf{s}), \quad \mathbf{s} &= (s_1, s_2) \\
(\mathbf{t}), \quad \mathbf{t} &= (t_1, t_2) \\
\frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2}\n\end{aligned}$   $\begin{aligned}\n\mathbf{\hat{s}}_j \circ \mathbf{F}^{(\ell)-1} \big) (\mathbf{x}) \longrightarrow \begin{bmatrix} \mathbf{\hat{A}}_j \\ \mathbf$ **r system**<br>
metrized with M patche<br>  $J^2 \rightarrow \mathbb{R}^3$ <br>
(s),  $\mathbf{s} = (s_1, s_2)$ <br>
(t),  $\mathbf{t} = (t_1, t_2)$ <br>  $\frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2}$ <br>  $\hat{\mathbf{s}}_j \circ \mathbf{F}^{(\ell)-1}$ )(**x**) <br>  $\mathbf{s}$  are of type<br>  $\frac{\partial}{\partial \mathbf{r}_j}$ ( )  $\mathbf{H} = [0,1]^2 -$ <br>  $\mathbf{F}^{(\ell)}(\mathbf{s}),$ <br>  $\mathbf{F}^{(\ell)}(\mathbf{s}),$ <br>  $\mathbf{F}^{(\ell)}(\mathbf{t}),$ <br>  $\mathbf{F}^{(\ell)} = \begin{vmatrix} \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} & \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2} \\ \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} & \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2} \\ \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} & \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_$ *x F s s* **if the Same of t Example 18 Assume that Assume the Set of Assume the Set of**  $\vec{F} = [0,1]^2 \rightarrow \mathbb{R}^3$ **<br>**  $\vec{F}^{(t)}(\mathbf{s})$ **,**  $\mathbf{s} = (s_1,s_2)$ **<br>**  $\vec{F}^{(t)}(\mathbf{t})$ **,**  $\vec{t} = (t_1,t_2)$ **<br>**  $\vec{f}$ **)**  $:= \left\| \frac{\partial \vec{F}^{(t)}}{\partial t_1} \times \frac{\partial \vec{F}^{(t)}}{\partial t_2} \right\|$ **<br>** 2 3  $\begin{CD} \mathbf{em} \ \mathbf{h} \mathsf{M} \ \mathsf{patches} \ \mathbf{h} \ \mathbf{h} \ \mathsf{M} \ \mathsf{patches} \ \mathbf{h} \end{CD}$  $\begin{CD} 1 & \text{if } \mathsf{R} \setminus \mathsf{R} \setminus$ System<br>trized with M pa<br>  $\rightarrow \mathbb{R}^3$ <br>  $\rightarrow \mathbb{R}^3$ <br>  $\rightarrow t = (t_1, t_2)$ <br>  $\rightarrow t = (t_1, t_2)$ <br>  $\rightarrow t \rightarrow t_1$ <br>  $\rightarrow t_2$ <br>  $\rightarrow t$ <br>  $\rightarrow t_1$ <br>  $\rightarrow t_2$ <br>  $\rightarrow t_1$ ar syster<br>
rametrized with M<br>
0,1]<sup>2</sup> → R<sup>3</sup><br>  $\mathbf{F}^{(\ell)}(\mathbf{s}), \quad \mathbf{s} = (s_1, s_2)$ <br>  $\mathbf{F}^{(\ell)}(\mathbf{t}), \quad \mathbf{t} = (t_1, t_2)$ <br>  $\mathbf{F}^{(\ell)}(\mathbf{t}), \quad \mathbf{t} = (t_1, t_2)$ <br>  $\mathbf{F}^{(\ell)}(\mathbf{t}) = \left\| \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2$ **e**<br>*n* M<br>*,s*<sub>2</sub>)<br>**e**<br>S<sub>*i*</sub><sup>(ℓ)</sup> **e**<br> *n N*<br> *,,s*<sub>2</sub><br> *,t*<sub>2</sub>)<br> *e*<br>
(*S*<sub>*i*</sub><sup>(ℓ</sup>  $\mathbf{x}^{(\ell)} = \mathbf{F}^{(\ell)}(\mathbf{s}), \quad \mathbf{s} = (s_1, s_2)$ <br>  $\mathbf{y}^{(\ell)} = \mathbf{F}^{(\ell)}(\mathbf{t}), \quad \mathbf{t} = (t_1, t_2)$  $J^{(\ell)}(t) := \left\| \frac{\partial F^{(\ell)}}{\partial t} \times \frac{\partial F^{(\ell)}}{\partial t} \right\|$ **system**<br>
terized with M pa<br>  $\rightarrow \mathbb{R}^3$ <br> *f* = ( $s_1$ , $s_2$ )<br> *f* = ( $t_1$ , $t_2$ )<br>  $\frac{1}{t_1} \times \frac{\partial F^{(\ell)}}{\partial t_2}$ <br>  $\rightarrow F^{(\ell)-1}$ )(**x**)<br>
are of type<br>  $\frac{\partial}{\partial t_1} G_K(\mathbf{s}_i^{(\ell)}, F)$ <br>
are of type<br>  $\frac{\partial}{\partial n_y} G_K(\mathbf{s}_i^{(\ell)}, F)$ <br>
a **The linear system**<br>
• Domain is parametrized with M patcl<br>  $F^{(\ell)} = [0,1]^2 \rightarrow \mathbb{R}^3$ <br>  $\mathbf{x}^{(\ell)} = F^{(\ell)}(\mathbf{s}), \quad \mathbf{s} = (s_1, s_2)$ <br>  $\mathbf{y}^{(\ell)} = F^{(\ell)}(\mathbf{t}), \quad \mathbf{t} = (t_1, t_2)$ <br>  $J^{(\ell)}(\mathbf{t}) := \left\| \frac{\partial F^{(\ell)}}{\partial t_1} \times \frac{\partial F^{(\ell)}}{\partial t_2} \$ **The linear system**<br>
Domain is parametrized with M patches:<br>  $F^{(\ell)} = [0,1]^2 \rightarrow \mathbb{R}^3$ <br>  $\mathbf{x}^{(\ell)} = F^{(\ell)}(\mathbf{s}), \quad \mathbf{s} = (s_1, s_2)$ <br>  $\mathbf{y}^{(\ell)} = F^{(\ell)}(\mathbf{t}), \quad \mathbf{t} = (t_1, t_2)$ <br>  $J^{(\ell)}(\mathbf{t}) := \left\| \frac{\partial F^{(\ell)}}{\partial t_1} \times \frac{\partial F^{(\ell)}}{\partial t_2}$ **i h c linear system**<br> **e** Domain is parametrized with M patches:<br>  $F^{(i)} = [0,1]^2 \rightarrow \mathbb{R}^3$  geometry mapping<br>  $\mathbf{x}^{(i)} = F^{(i)}(\mathbf{s}), \quad \mathbf{s} = (s_1, s_2)$  collocation point<br>  $\mathbf{y}^{(i)} = F^{(i)}(\mathbf{t}), \quad \mathbf{t} = (t_1, t_2)$  int **is parametrized with M patches:**<br>  ${}^{(c)} = [0,1]^2 \rightarrow \mathbb{R}^3$  geometry mapping<br>  ${}^{(c)} = F^{(c)}(s)$ ,  $s = (s_1, s_2)$  collocation point<br>  ${}^{(c)} = F^{(c)}(t)$ ,  $t = (t_1, t_2)$  integration point<br>  ${}^{(i)}(t) := \left\| \frac{\partial F^{(i)}}{\partial t_1} \times \frac{\partial F^{(i)}}{\$ r Syst<br>
netrized wi<br>  $J^2 \rightarrow \mathbb{R}^3$ <br>
(s),  $\mathbf{s} = (s$ <br>
(t),  $\mathbf{t} = (t \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_2} \times \frac{\partial \mathbf{F}^{(\ell)}}{\partial t_1$ **EXAMPLE SAMPLE SET ASSES SET ASSESS SET ASSES AND ASSESS AND SET ASSESS AND SET AND ASSESS AND SET ASSESS AND SERVED AS A SET AND AND SERVED ASSESS AND SERVED AND SERVED AND SERVED AND SERVED AND SERVED AND SERVED ASSESS arametrized with M patches:**<br>  $\begin{array}{lll} \epsilon[\mathbf{0},1]^2\to\mathbb{R}^3 & \text{geometry} \ \epsilon[\mathbf{0},1]^2\to\mathbb{R}^3 & \text{geometry} \ \epsilon[\mathbf{0},1]^2\to\mathbb{R}^3 & \text{clocatile} \ \epsilon[\mathbf{0},1]^2\to\mathbb{R}^3 & \text{collocatile} \ \epsilon[\mathbf{0}](t), & \textbf{t}=(t_1,t_2) & \text{integral} \ \epsilon[\mathbf{0}(t), & \textbf{t}=(t_1,t_2) & \text{surface$ 

*geometry mapping collocation point integration point t*<sub>2</sub>) integration point<br> *surface area element* tches:<br>
geometry mapping<br>
collocation point<br>
integration point<br>
surface area element<br>  $\begin{aligned} \mathbf{A} \ \pmb{\alpha} & = \pmb{\beta} \end{aligned} \quad \mathbf{A} : \mathbf{M} \times \mathbf{M} \text{ blc} \ \begin{aligned} & ^{(k)}(\pmb{t})) \ \hat{B}^{(k)}_{j,d}(\pmb{t}) \ J^{(k)}(\pmb{t}) d\pmb{t} + \frac{1}{2} \ \end{aligned} \ \begin{$ 

 $A_0(X) = \sum_i \sum_i \alpha_i^X$  ( $D_i \circ F$  )  $(X)$   $\longrightarrow$   $|A\setminus G| = |P|$  A: M x M block matrix **EXECUTE:**<br>
Determined with M patches:<br>  $=[0,1]^2 \rightarrow \mathbb{R}^3$  geometry mapping<br>  $= F^{(i)}(s)$ ,  $s = (s_1, s_2)$  collocation point<br>  $= F^{(i)}(t)$ ,  $t = (t_1, t_2)$  integration point<br>  $t) := \left\| \frac{\partial F^{(i)}}{\partial t_1} \times \frac{\partial F^{(i)}}{\partial t_2} \right\|$  surfac **Car System**<br>
parametrized with M patches:<br>  $=[0,1]^2 \rightarrow \mathbb{R}^3$  geometry mapping<br>  $=F^{(t)}(\mathbf{s}), \quad \mathbf{s}=(s_1,s_2)$  collocation point<br>  $\mathbf{r} = F^{(t)}(\mathbf{r}), \quad t = (t_1,t_2)$  integration point<br>  $\mathbf{r} = \left| \frac{\partial F^{(t)}}{\partial t_1} \times \frac{\partial F^{(t)}}{\partial t_$  $=1$  jeJ  $=\sum\sum\alpha_{\bm{j}}^{(\ell)}~(\hat{\bm{B}}_{j}\circ \bm{\digamma}^{(\ell)-1})(\bm{x})$  and  $\mathbf{j} \in \mathcal{J}$ *M*

**• The matrix entries are of type** 

linear system

\nn is parametrized with M patches:

\n
$$
F^{(t)} = [0,1]^2 \rightarrow \mathbb{R}^3
$$
\n
$$
x^{(t)} = F^{(t)}(s), \quad s = (s_1, s_2)
$$
\n
$$
y^{(t)} = F^{(t)}(t), \quad t = (t_1, t_2)
$$
\n
$$
y^{(t)} = \left\| \frac{\partial F^{(t)}}{\partial t_1} \times \frac{\partial F^{(t)}}{\partial t_2} \right\|
$$
\n
$$
y^{(t)} = \left\| \frac{\partial F^{(t)}}{\partial t_1} \times \frac{\partial F^{(t)}}{\partial t_2} \right\|
$$
\n
$$
\sum_{j \in \mathcal{J}} \alpha_j^{(t)} (\hat{B}_j \circ F^{(t-1)})(x)
$$
\n
$$
\Rightarrow \text{A. } \alpha = \beta \quad \text{A. } \alpha \times \text{M block matrix}
$$
\nmatrix entries are of type

\n
$$
\mathbb{A}_{ij}^{(t,k)} = \int_{[0,1]^2} \frac{\partial}{\partial n_y} \mathcal{G}_k(s_i^{(t)}, F^{(k)}(t)) \hat{B}_{j,d}^{(k)}(t) J^{(k)}(t) dt + \frac{1}{2} \hat{B}_{j,d}^{(k)}(s_i^{(t)})
$$
\nthat hand side entries:

\n
$$
\beta_{ij}^{(t)} = \sum_{k=1}^{M} \int_{[0,1]^2} \mathcal{G}_k(s_i^{(t)}, F^{(k)}(t)) u_N(F^{(k)}(t)) J^{(k)}(t) dt
$$
\nIt has simple terms to solve

\nHamilton's function

• The right hand side entries:

$$
\boldsymbol{\beta}_{ij}^{(\ell)} = \sum_{k=1}^M \int_{[0,1]^2} \mathcal{G}_{k}(\boldsymbol{s}_i^{(\ell)}, \boldsymbol{F}^{(k)}(t)) u_N\left(\boldsymbol{F}^{(k)}(t)\right) J^{(k)}(t) dt
$$



## IGA-BEM pipeline

#### Discretization of the surface Γ

- Multi-patch parametric representation by tensor product splines (B-splines or NURBS)
- patch topology conforming meshes ( $\mathcal{C}^{-1}$  or  $\mathcal{C}^0)$
- Discretization of the BIE (N= #DoF)
	- **o** Collocation method

**[Degli Esposti, Falini, Kanduc, S, Sestini,** CAMWA**, 2024]**

- Construction and solution of the linear system
	- **•** regular, singular and near-singular quadrature based on the spline product formula and quasi-interpolation. Quadrature always developed on B-spline supports
	- non-symmetric and fully-populated matrix

#### Representation formula to evaluate quantities in the exterior domain

● cost reduced to a matrix/vector multiplication

#### **Limitations of standard IGA-BEM**

- $\bullet$  to improve accuracy we have to use finer meshes  $\rightarrow$  high costs in terms of memory and CPU time
- limited geometric complexity and frequency range (due to the size of the final linear system)



### Fast solvers for IGA-BEM

- Need of an efficient approximate method to evaluate the matrix entries, that allows to define a fast solver
- $\bullet$  Hierarchical matrices, or  $\mathcal H$ -matrices, have been introduced in the BEM setting by Hackbusch as a technique to produce sparse-data representation of dense matrices, which carries improvements in terms of storage and computational cost with respect to the usual matrix operations **[Hackbusch,** *Computing***, 1993]**

H-matrices:

- $\bullet$  representation of the BEM matrix with an  $H$ -matrix structure
- reduction of the memory cost: low-rank approximation of large blocks
- $\bullet$  optimization of the CPU times by using the  $H$ -matrix/vector product
- Pure algebraic approach
- Alternative approach to Fast Multipole Method **[Greengard & Rokhlin,** *J. Comp. Phys***, 1987]**
	- For the Helmholtz problem a diagonal FFM has been developed **[Rokhlin,** *Appl. Comp. Harm***, 1993]**
	- **•** Different formulations for low and high frequencies



# Hierarchical clustering of DoF

#### $\bullet$   $\mathcal{H}$ -matrix representation of the system matrix

Preliminary clustering of row and columns based on the geometry (physical distance): definition of two Binary Trees,  $\mathcal{T}^{(\ell)}_1$  and  $\mathcal{T}^{(\ell)}_2$  whose depth is determined by a parameter n leaf

 $\bullet$  rows  $\leftrightarrow$  collocation points =reference points

• columns  $\leftrightarrow$  basis support  $\leftrightarrow$  basis referred points=reference points

 $\tau^{(\ell)}_{i}$  i=1,2

- $\bullet$  initial box : bounding box of the patch reference points
- dyadic subdivision into balanced small boxes
- **o** stop subdivision when a minimum number of points per box is reached
- Remark: negligible cost
- In total we have to construct 2M cluster trees
- The interaction of any two of these cluster trees form a block cluster tree



## Cluster tree algorithm

Subdivision of the IgA-BEM matrix (Block Cluster Tree): definition of submatrices, again Euclidean distance based

We have to define a criterion the determine whether a block has a suitable *lowrank approximation:*

- The block should be as large as possible
- Computing explicitly the rank of the blocks is too expensive

A block associated to the cluster indices  $(\sigma, \tau)$ , with  $\sigma \in \mathcal{T}_1^{(\ell)}$  and  $\tau \in \mathcal{T}_2^{(\ell)}$  is admissible if  $min(diam(Q_{\sigma}), diam(Q_{\tau})) \leq \eta \text{ dist}(Q_{\sigma}, Q_{\tau})$ 





3 kinds of blocks: leaves (full- or low-rank matrices) and non-leaves (H-matrices)

# Structure of the H-matrix representation

The structure of the H-matrix representation of the BEM matrix depends only on the geometry and the admissibility condition







full-rank block low-rank block





#### Low-rank representation

The reduction of the memory storage of the IgA-BEM is related to the possibility of writing a *low-rank representation* or degenerate expansion of the fundamental solution  $\mathcal{G}_{\kappa}$ , i.e. **PRESENT ATTER EXECTS AND REFERENT ATTENTS ATTACK ATTENDANT AND ATTENTIFY ATTENTIFY (***x,y)* **=**  $\sum_{k=0}^{L} \varphi_k(\mathbf{x}) \psi_k(\mathbf{y}) + R_r(\mathbf{x}, \mathbf{y})$ **,<br>duum and tends to zero for**  $r \to \infty$  **[Bebendorf, 2008]<br>it can be proved that if th entation**<br>
orage of the IgA-BEM is related to the possibility of<br>
or degenerate expansion of the fundamental<br>  $=\sum_{k=0}^{r} \varphi_k(\mathbf{x}) \psi_k(\mathbf{y}) + R_r(\mathbf{x}, \mathbf{y}),$ <br>
and tends to zero for  $r \to \infty$  [Bebendorf, 2008]<br>
be proved that **tation**<br> **c** of the IgA-BEM is re<br> **egenerate expansion**<br>  $\psi_k(\mathbf{x}) \psi_k(\mathbf{y}) + R_r(\mathbf{x}, \mathbf{y})$ <br> **ends to zero for**  $r \to \infty$ *x y x y x y R* eduction of the men<br>
a *low-rank represe*<br>
n  $G_{\kappa}$ , i.e.<br>  $G$ <br>  $R_r(\mathbf{x}, \mathbf{y})$  is the resi **[Chaillat, Desiderio, Ciarlet, J. Comp. Phys. 2017]<br>
<b>Exerces and Conservant Conserved Expansion** of the fundamental<br>  $\mathcal{G}_x(x, y) = \sum_{k=0}^{L} \varphi_k(x) \psi_k(y) + R_r(x, y)$ ,<br>
the residuum and tends to zero for  $r \to \infty$  [Babendorf, 2 **9 | gA-BEM** is related to the possibility of<br>
rate expansion of the fundamental<br>  $\chi(\mathbf{y}) + R_r(\mathbf{x}, \mathbf{y}),$ <br> **9 zero for**  $r \to \infty$  [Bebendorf, 2008]<br>
nat if the admissibility condition is<br>
m above.<br>  $\gamma = \frac{1 + \kappa ||\mathbf{x} - \mathbf{$ related to the possibility of<br>
on of the fundamental<br>
y),<br>
→ ∞ [Bebendorf, 2008]<br>
issibility condition is<br>  $+ \kappa ||\mathbf{x} - \mathbf{y}||$ <br>  $C_2$ <br>
--asymptotic regime for<br>
iderio, Ciarlet, J. Comp. Phys, 2017] **DIFESENTATION**<br>
mory storage of the IgA-BEM is related to the possibility of<br>
entation or degenerate expansion of the fundamental<br>
iduum and tends to zero for  $r \rightarrow \infty$  [Bebendort, 2008]<br>
it can be proved that if the admi **Example 18 and the possible of the fundamenta**<br> **Example 18 and 18 Propressertation** or<br>  $g_x(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{r} g_k(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{r}$ <br> **c** residuum and<br>
small  $\rightarrow \gamma \cong \frac{c_1}{c_2}$ <br>  $\downarrow \downarrow \downarrow$   $\downarrow$   $\downarrow$   $\uparrow$   $\parallel$ <br>  $\downarrow$  small  $\rightarrow \gamma \cong \frac{1}{c_2}$ .<br>
epresentation is **Sentation**<br>storage of the IgA<br>*ion* or degenerate<br> $\gamma$ ) =  $\sum_{k=0}^{r} \varphi_k(\mathbf{x}) \psi_k(\mathbf{y})$ <br>m and tends to ze<br>n be proved that if<br>bounded from at<br> $\frac{c_1}{\mathbf{x} - \mathbf{y}} \frac{(\sqrt{3} \gamma \eta)^r}{1 - \sqrt{3} \gamma \eta}$ <br> $\gamma \approx \frac{1}{c_2}$ . Existence<br>t  $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br>  $\frac{1}{2}$ <br>  $\frac{1}{\sqrt{3}}$ <br>  $\frac{1}{$ 

$$
\mathcal{G}_{\kappa}(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{r} \varphi_{k}(\mathbf{x}) \psi_{k}(\mathbf{y}) + R_{r}(\mathbf{x}, \mathbf{y}),
$$

where  $R_c(\mathbf{x}, \mathbf{y})$  is the residuum and tends to zero for  $r \to \infty$  [Bebendorf, 2008]

For the Helmholtz kernel it can be proved that if the admissibility condition is satisfied, the residuum can be bounded from above.

$$
|R_r(\mathbf{x}, \mathbf{y})| \le \frac{C_1}{C_2 \left\| \mathbf{x} - \mathbf{y} \right\|^m} \frac{(\sqrt{3} \gamma \eta)^r}{1 - \sqrt{3} \gamma \eta} \qquad \gamma = \frac{1 + \kappa \left\| \mathbf{x} - \mathbf{y} \right\|}{C_2}
$$

When  $\kappa$  diam( $\Gamma$ ) is small  $\rightarrow \gamma \cong \frac{1}{\epsilon}$  $c<sub>2</sub>$ . Existence of a a pre-asymptotic regime for which the low-rank representation is efficient



## Computation of low-rank representation

Given an admissible block  $\mathbb{A}_{\sigma,\tau}^{\{\ell,\tau\}}$  $(\ell,\overline{\ell})$  $\in \mathcal{C}^{m \times n}$  we approximate it as the product of matrices of small rank ( , ) **COM-rank reproduce the CAST COM-rank reproduce the CAST CONTEXT**<br>  $\frac{\overline{(l,\bar{l})}}{\sigma,\tau} \in C^{m \times n}$  we approximate<br>  $\frac{\overline{(l,\bar{l})}}{\sigma,\tau} = \mathbb{S}_r + \mathbb{R}_r$ <br>
are both  $N \times r$  matrices and  $DW$ -rank representation<br>  $\in C^{m \times n}$  we approximate it as the product of<br>  $=\mathbb{S}_r + \mathbb{R}_r$ <br>
a both  $N \times r$  matrices and the residuum  $\mathbb{R}_r$  is

with  $\mathbb{S}_r = \mathbb{U} \; \mathbb{V}^H$  where  $\mathbb{U}$  and  $\mathbb{V}$  are both  $N \times r$  matrices and the residuum  $\mathbb{R}_r$  is such that **of low-rank represen**<br>  $ck \mathbb{A}_{\sigma,\tau}^{(\ell,\bar{\ell})} \in C^{m \times n}$  we approximate it as the<br>  $\mathbb{A}_{\sigma,\tau}^{(\ell,\bar{\ell})} = \mathbb{S}_r + \mathbb{R}_r$ <br>
and V are both  $N \times r$  matrices and the res<br>  $\mathbb{A}_{\sigma,\tau}^{(\ell,\bar{\ell})} - \mathbb{S}_r \Big|_F = \left\| \mathbb{A}_{\sigma,\tau}^{(\ell,\$ **of low-rank representation**<br>  $\alpha \in \mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} \in C^{m \times n}$  we approximate it as the product of<br>  $\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} = \mathbb{S}_r + \mathbb{R}_r$ <br>
and V are both  $N \times r$  matrices and the residuum  $\mathbb{R}_r$  is<br>  $\mathbb{A}_{\sigma,\tau}^{(\$ **ON Of low-rank representation**<br>
⇒ block  $\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} \in \mathbb{C}^{m \times n}$  we approximate it as the product of<br>
nk<br>  $\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} = \mathbb{S}_r + \mathbb{R}_r$ <br>
⇒ U and V are both  $N \times r$  matrices and the residuum  $\mathbb{R}_$  $\|\mathbb{R}_r\|_F = \|\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} - \mathbb{S}_r\|_F = \|\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} - \mathbb{U}\,\mathbb{V}^H\|_F \leq \varepsilon \|\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})}\|_F$ **ation of low-rank represe**<br>
sible block  $\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} \in C^{m \times n}$  we approximate it as t<br>
If rank  $\boxed{\mathbb{A}_{\sigma,\tau}^{(\ell,\overline{\ell})} = \mathbb{S}_r + \mathbb{R}_r}$ <br>
here U and V are both  $N \times r$  matrices and the r<br>  $r \Vert_F = \left\| \mathbb{A}_{\sigma,\tau}^{$ 

•r 
$$
<< N
$$
 we obtain a **drastic reduction** of the memory requirement for the storage of  $A_{σ,τ}^{(\ell, F)} \Rightarrow$  how to compute ℤ and ℤ ?

- The best low-rank approximation is given by the truncated SVD. Its computation is too expensive as it requires in input all the entries of the matrix.
- Adaptive Cross Approximation (ACA) produces quasi-optimal low-rank approximations without requiring the assembly of the whole matrix
	- Every matrix of rank r is the sum of r matrices of rank 1
	- Greedy algorithm iteratively adding suitable 1-rank matrices to the current approximation
	- ್ಷ**್ಡಾ**Requires only few entries of the matrix



**[Babendorf,Rjasanow,** Computing, **2003]**

## Example 1: rigid scattering on a sphere

- $\Omega$  =domain exterior to a sphere centered at the origin and with unit radius
- Acoustic pressure produced by a wave vector, **O** source at infinity



**1: rigid scattering on a**<br> *i* or to a sphere centered at the origin and<br> *v* produced by a wave vector,<br>  $p_{inc} = e^{i\kappa(\mathbf{v} \cdot \mathbf{x})}$   $\mathbf{v} = (1,0,0)$ <br> *v* =  $(1,0,0)$ <br> *v* =  $i\hbar \mathbf{v}$ <br> *v* =  $i\hbar \mathbf{v}$ <br> *v* =  $i\hbar$ The incident wave hitting the rigid body produces a scattered  $\bullet$ pressure

**mple 1: rigid scattering on a sphere**  
\nmain exterior to a sphere centered at the origin and with unit radius  
\nc pressure produced by a wave vector,  
\nat infinity 
$$
p_{inc} = e^{i\kappa(\mathbf{v} \cdot \mathbf{x})}
$$
  $\mathbf{v} = (1,0,0)$   
\nident wave hitting the rigid body produces a scattered  
\n  
\n  
\n
$$
\begin{cases}\n\Delta p_s + \kappa^2 p_s = 0 & \text{in } \Omega \\
\frac{\partial_\rho p_s}{\partial_n p_s} = -\frac{\partial_\rho p_{inc}}{\partial_n p_{inc}} & \text{on } \Gamma \\
\frac{\partial_\rho p_s}{\partial_n p_s} = -\frac{\partial_\rho p_{inc}}{\partial_n p_{inc}} + p_s & \text{on } \Gamma\n\end{cases}
$$
\n
$$
p_{tot} = p_{inc} + p_s
$$
\n
$$
p_s(\mathbf{x}) = \sum_{n=0}^{\infty} \frac{i^n (2n+1) j'( \kappa R)}{h'_n (\kappa R)} p_n ( \cos(\theta) ) h_n (\kappa R) \qquad n = 10
$$

$$
\boldsymbol{p}_{tot} = \boldsymbol{p}_{inc} + \boldsymbol{p}_s
$$

$$
p_{s}(x) = \sum_{n=0}^{\infty} \frac{i^{n} (2n+1) j'(kR)}{h'_{n}(kR)} P_{n}(cos(\theta)) h_{n}(kR) \qquad n = 10
$$



M. Lucia Sampol

### Rigid scattering on a sphere (low- freq)

 $\kappa = 1$  $n_{leaf}$  = 25  $\eta = 3$  $\varepsilon_{ACA} = 1.0e - 08$  $\varepsilon_{GMRES} = 1.0e - 08$ 

 $\kappa$  diam( $\Gamma$ ) <  $2\pi$ *Low-freq.*







An iterative solver (variant of GMRES) for  $H$ -matrices is considered.  $\bullet$ 



H-matrices for 3D IGABEM 19

## Rigid scattering on a sphere (high- freq)

 $\kappa = 3$  $n_{leaf}$  = 25  $\eta = 3$  $\varepsilon_{ACA} = 1.0e - 08$  $\varepsilon_{GMRES} = 1.0e - 08$ 





#### $\kappa$  diam( $\Gamma$ ) >  $2\pi$ *High -freq.*





Very good accuracy for engineering applications

### Rigid scattering on a sphere

Reconstructed total field,  $N_{\text{dof}}$ =2904  $\bullet$ 



#### real part imaginary part



# Example 2: acoustic problem (interior)

Helmholtz problem interior to a torus, Neumann conditions with exact solution



## Example 2: acoustic problem (interior)



#### $\kappa =$  $\varepsilon$ GMRES

 $\kappa = 3$ 

 $\varepsilon_{GMRRS} = 1.0e - 08$ 

N<sub>DOF</sub> mem (Δ) and mem (β) and mem (m) and  $\rm{Err}$ 864 −41.05% −37.03% 1.15e − 02 2904 21.63% 19.52% 9.91e − 04 10584 67.92% 69.06% 1.07e − 04 40344 89.11% 89.81% 1.29e − 05 157464 96.57% 97.11% 1.65e − 06



### **Conclusion**

- An efficient and accurate numerical strategy to solve 3D Helmholtz problems using isogeometric BEMs on conformal multi-patch smooth geometries and spline discretization spaces by hierarchical matrices.
- It gives good results achieving optimal approximation order with a drastic reduction of the computational cost, in terms of both time and memory requirement.

#### **•** Future work:

- Helmholtz equation for different  $\kappa$  (special treatment of highly oscillating singular integrals coming from IgA-BEM)
- **•** HPC implementation
- **Time-domain problems**

## Thank you for your attention



# **SMART 2025**

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