A link between Chebyshev polynomials and level-dependent subdivision schemes

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Goal

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Subdivision schemes **Chebyshev polynomials**

We create a link between the two topics which is exploited to fill theoretical gaps and improve computational issues related to level-dependent subdivision schemes.

Goal and Outline of the talk

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Subdivision schemes  **Chebyshev polynomials**

We create a link between the two topics which is exploited to fill theoretical gaps and improve computational issues related to level-dependent subdivision schemes.

Outline

1. Preliminaries and state-of-the-art
2. New theoretical results
3. Practical implications

Univariate binary level-dependent subdivision schemes

UNivAriate bInary leVel-dEpendent (NAIVE*) subdivision schemes

*(this is only an easy-to-remember abbreviation!)

Univariate binary subdivision schemes: the functional case

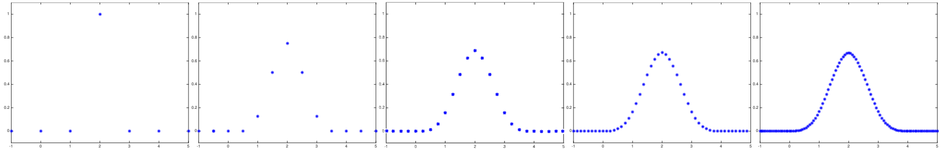
Let $\mathbf{P}_0 = \{P_0(i) \in \mathbb{R} \ : \ i \in \mathbb{Z}\}$ be an initial sequence of points attached to the integer grid. For $k \geq 0$, the subdivision scheme computes the sequence

$$\mathbf{P}_{k+1} := S_{\mathbf{a}} \mathbf{P}_k$$

via the subdivision operator

$$S_{\mathbf{a}} : \ell(\mathbb{Z}) \to \ell(\mathbb{Z}), \qquad (S_{\mathbf{a}} \mathbf{P}_k)(i) = \sum_{j \in \mathbb{Z}} a(i - 2j) \ P_k(j), \quad i \in \mathbb{Z},$$

where $\ell(\mathbb{Z})$ is the space of sequences indexed by $\mathbb{Z}$.



$k = 0$ $\qquad$ $k = 1$ $\qquad$ $k = 2$ $\qquad$ $k = 3$ $\qquad$ $k = 4$

Univariate binary subdivision schemes: the functional case

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where $\ell(\mathbb{Z})$ is the space of sequences indexed by $\mathbb{Z}$.

☞ Binary subdivision: at each iteration the information is "doubled"

☞ $\boxed{\mathbf{a} = \{a(i) \in \mathbb{R} \ : \ i \in \mathbb{Z}\}}$ *subdivision mask*

Univariate binary level-dependent ("NAIVE") subdivision

Subdivision with a different set of coefficients at each level:

$$\{S_{\mathbf{a}_k},\ k \geq 0\} \quad \Leftrightarrow \quad \begin{cases} \text{Input:} \quad \mathbf{P}_0 \\ \text{For} \ \ k = 0, 1, \cdots \\ \qquad \mathbf{P}_{k+1} := S_{\mathbf{a}_k} \mathbf{P}_k \quad \text{level-dependent rules} \end{cases}$$

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Convergence (Definition)

The scheme $\{S_{\mathbf{a}_k},\ k \ge 0\}$ applied to the initial data $\mathbf{P}_0 \in \ell(\mathbb{Z})$ is called *convergent* if there exists a function $f_{\mathbf{P}_0} \in C(\mathbb{R})$, $f_{\mathbf{P}_0} \neq 0$, such that

$$\boxed{\lim_{k\to\infty} \sup_{i\in\mathbb{Z}} |\ f_{\mathbf{P}_0}(2^{-k} i) - \mathbf{P}_k(i)\ | = 0.}$$

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☞ We assume the scheme to be non-singular, i.e., to generate the zero function if and only if $\mathbf{P}_0$ is the zero sequence.

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Regularity (Definition)

The scheme $\{S_{\mathbf{a}_k},\ k \geq 0\}$ is *C^m−convergent* if $f_{\mathbf{P}_0} \in C^m(\mathbb{R})$.

How to prove convergence and other useful properties?

We can make use of standard mathematical tools of signal processing: the z-trasform or subdivision symbol associated with the subdivision mask.

Subdivision symbol (Definition)

The subdivision symbol of a subdivision mask $\mathbf{a} = \{a(i),\ i \in \mathbb{Z}\}$ is the Laurent polynomial

$$a(z) = \sum_{i \in \mathbb{Z}} a(i)\, z^i, \qquad z \in \mathbb{C} \setminus \{0\}.$$

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- In the level-dependent case, the sequence of symbols $\{a_k(z),\ k \geq 0\}$ associated with the sequence of masks $\{\mathbf{a}_k,\ k \geq 0\}$ identifies the subdivision scheme.
- Many of the properties of a subdivision scheme can be easily checked using algebraic conditions on the subdivision symbols.

Generation versus *Reproduction* of a function space $\mathcal{V}$

Generation/Reproduction (Definition)

- A convergent subdivision scheme $\{S_{\mathbf{a}_k},\, k \geq 0\}$ generates $\mathcal{V}$ if for any $f \in \mathcal{V}$ there exists $\mathbf{P}_0$ s.t. $\lim_{k\to\infty} S_{\mathbf{a}_k} S_{\mathbf{a}_{k-1}} \ldots S_{\mathbf{a}_0} \mathbf{P}_0 = f$.

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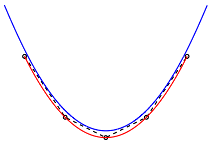
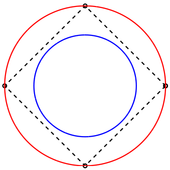
- A convergent subdivision scheme $\{S_{\mathbf{a}_k},\ k \geq 0\}$ reproduces $\mathcal{V}$ (with respect to $\mathbf{t}_0$) if for any $f \in \mathcal{V}$ and $\mathbf{P}_0 = \{f\left(t_0(i)\right),\ i \in \mathbb{Z}\}$ we have $$\lim_{k\to\infty} S_{\mathbf{a}_k} S_{\mathbf{a}_{k-1}} \dots S_{\mathbf{a}_0} \mathbf{P}_0 = f.$$





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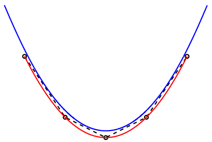

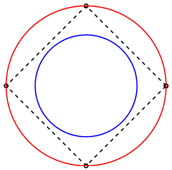

☞ The **approximation order** of a subdivision scheme is strictly related to its reproduction properties.

Exponential-Polynomial generation versus reproduction

Exponential-polynomials (Definition)

Let $n \in \mathbb{N}$ and let $\Gamma = \{(\theta_1, \xi_1), \ldots, (\theta_n, \xi_n)\}$ with $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$, $\theta_i \neq \theta_j$ if $i \neq j$, and $\xi_i \in \mathbb{N}$, $i = 1, \cdots, n$. The function space

$$EP_\Gamma = \mathrm{span}\{\ x^{r_i}\, e^{\theta_i x},\ r_i = 0, \cdots, \xi_i - 1,\ i = 1, \cdots, n\ \}$$

is a very general space of exponential polynomials.

For each $i = 1, \cdots, n$, ξ_i denotes the multiplicity of $\theta_i \in \mathbb{R} \cup \mathrm{i}\mathbb{R}$.

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☞ Generation and reproduction of EP_Γ for any subdivision scheme $\{S_{\mathbf{a}_k},\ k \geq 0\}$ can be easily checked by using simple algebraic conditions on the subdivision symbols

$$a_k(z) = \sum_{i \in \mathbb{Z}} a_k(i)\, z^i, \qquad z \in \mathbb{C} \setminus \{0\}, \qquad k \geq 0.$$

Algebraic conditions for exponential-polynomial gen./rep.

Theorem [C. Conti and L. R. 2011]

Let $n \in \mathbb{N}$, $\Gamma = \{(\theta_1, \xi_1), \cdots, (\theta_n, \xi_n)\}$, $z_{\ell,k} = e^{\frac{-\theta_\ell}{2^{k+1}}}$, $\ell = 1, \cdots, n$, $k \geq 0$.

A non-singular, convergent subdivision scheme $\{S_{\mathbf{a}_k},\ k \geq 0\}$

- generates EP_Γ iff $\dfrac{d^r\, a_k(-z_{\ell,k})}{dz^r} = 0, \ r = 0, ..., \xi_\ell - 1 \qquad (*)$
- reproduces EP_Γ with respect to $\mathbf{t}_k = \{\frac{i+\tau}{2^k},\ i \in \mathbb{Z}\}$ iff, besides $(*)$,

  $a_k(z_{\ell,k}) = 2\,(z_{\ell,k})^\tau$ and

  $$\frac{d^r\, a_k(z_{\ell,k})}{dz^r} = 2\,(z_{\ell,k})^{\tau - r} \prod_{q=0}^{r-1} (\tau - q), \quad r = 1, ..., \xi_\ell - 1.$$

C. Conti, L. R., Algebraic conditions on non-stationary subdivision symbols for exponential polynomial reproduction, *J. Comput. Appl. Math.*, 236(4), 543-556, (2011)

The EP_Γ space with integer powers of exponentials

Let $n \in \mathbb{N}$, $t \in [0, \pi/n) \cup \mathrm{i}\mathbb{R}^+$ and $\Gamma = \{(\theta_1, \xi_1), \cdots, (\theta_{2n+1}, \xi_{2n+1})\}$ with

$$\begin{array}{llll} \theta_1 = 0, & \theta_{2\ell} = \mathrm{i}\,\ell t, & \theta_{2\ell+1} = -\mathrm{i}\,\ell t, & \ell = 1, ..., n \\ \xi_1 = 2, & \xi_{2\ell} = 1, & \xi_{2\ell+1} = 1, & \ell = 1, ..., n. \end{array}$$

Then

$$EP_\Gamma^{2n+2} = \mathrm{span}\left\{ 1, x, \{e^{\mathrm{i}\ell t x}, e^{-\mathrm{i}\ell t x}\}_{\ell=1}^n \right\}$$

and, in particular,

$$EP_\Gamma^{2n+2} = \begin{cases} \mathrm{span}\left\{ 1, x, \{\cos(\ell\sigma x), \sin(\ell\sigma x)\}_{\ell=1}^n \right\} & \text{if } t = \sigma,\ \sigma \in (0, \pi/n) \\ \mathrm{span}\left\{ 1, x, x^2, \ldots, x^{2n+1} \right\} & \text{if } t = 0 \\ \mathrm{span}\left\{ 1, x, \{\cosh(\ell\sigma x), \sinh(\ell\sigma x)\}_{\ell=1}^n \right\} & \text{if } t = \mathrm{i}\sigma,\ \sigma \in \mathbb{R}^+ \end{cases}$$

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Goal:
express the symbols of the minimum support approximating scheme generating EP_Γ^{2n+2} and the minimum support interpolating scheme reproducing EP_Γ^{2n+2} in closed form.

State of the art

The exponential B-spline scheme generating EP_{Γ}^{2n+2} is known to be defined by the k-level symbol

$$s_{2n+2,k}(z) = \frac{(1+z)^2}{2z}\, z^{-n} \prod_{j=1}^{n} \frac{z^2+\lambda_j(v_k)z+1}{\lambda_j(v_k)+2}$$

with

$$v_k = \cos\left(\frac{t}{2^{k+1}}\right) \quad \text{and} \quad \lambda_j(v_k) = \det\left(\begin{bmatrix} 2v_k & 1 & 0 & 0 & \cdots & 0 \\ 2 & 2v_k & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2v_k & 1 & \cdots & \vdots \\ 0 & 0 & 1 & 2v_k & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2v_k \end{bmatrix}_{j\times j}\right)$$

☞ The larger n the higher the computational complexity!

L. R., From approximating subdivision schemes for exponential splines to high-performance interpolating algorithms, *J. Comput. Appl. Math.*, 224, 383-396, (2009)

From the B-spline scheme generating EP_Γ^{2n+2} to the interpolating one reproducing EP_Γ^{2n+2}

Input: $s_{2n+2,k}(z)$, k-level symbol of the approximating scheme generating EP_Γ^{2n+2}

- Construct the $(2n+1)\times(2n+1)$ matrix $\mathbf{H}_k$, leading principal submatrix of a Hurwitz type matrix associated to $s_{2n+2,k}(z)$
- Determine $(\mathbf{H}_k)^{-1}$
- Set $\mathbf{c}_k := (\mathbf{H}_k)^{-1}(n+1,:)$
- Construct the interpolatory symbol $m_{2n+2,k}(z) := s_{2n+2,k}(z)\ c_k(z)$

Output: $m_{2n+2,k}(z)$, k-level symbol of the interpolating scheme reproducing EP_Γ^{2n+2}

C. Conti, L. Gemignani, L. R., From approximating to interpolatory non-stationary subdivision schemes with the same generation properties, *Adv. Comput. Math.*, 35, 217-241, (2011)

Example ($n = 3$)

$$s_{8,k}(z) = \frac{(1+z)^2}{2z}\ z^{-n} \prod_{j=1}^{3} \frac{z^2 + \lambda_j(v_k)z + 1}{\lambda_j(v_k) + 2}$$

with

$$v_k = \cos\left(\frac{t}{2^{k+1}}\right)$$

and

$$\lambda_1(v_k) = 2v_k$$

$$\lambda_2(v_k) = \det\left(\begin{bmatrix} 2v_k & 1 \\ 2 & 2v_k \end{bmatrix}\right) = 4v_k^2 - 2$$

$$\lambda_3(v_k) = \det\left(\begin{bmatrix} 2v_k & 1 & 0 \\ 2 & 2v_k & 1 \\ 0 & 1 & 2v_k \end{bmatrix}\right) = 8v_k^3 - 6v_k$$

Shortly: $$s_{8,k}(z) = z^{-4}\left(s_0(v_k) + s_1(v_k)\,z + s_2(v_k)\,z^2 + \cdots\; s_8(v_k)z^8\right)$$

$$\mathbf{H}_k = \begin{bmatrix} s_1(v_k) & s_3(v_k) & s_5(v_k) & s_7(v_k) & 0 & 0 & 0 \\ s_0(v_k) & s_2(v_k) & s_4(v_k) & s_6(v_k) & s_8(v_k) & 0 & 0 \\ 0 & s_1(v_k) & s_3(v_k) & s_5(v_k) & s_7(v_k) & 0 & 0 \\ 0 & s_0(v_k) & s_2(v_k) & s_4(v_k) & s_6(v_k) & s_8(v_k) & 0 \\ 0 & 0 & s_1(v_k) & s_3(v_k) & s_5(v_k) & s_7(v_k) & 0 \\ 0 & 0 & s_0(v_k) & s_2(v_k) & s_4(v_k) & s_6(v_k) & s_8(v_k) \\ 0 & 0 & 0 & s_1(v_k) & s_3(v_k) & s_5(v_k) & s_7(v_k) \end{bmatrix}_{7\times 7}$$

This matrix must be inverted in order to determine $\mathbf{c}_k := (\mathbf{H}_k)^{-1}(4,:)$ and construct the interpolatory symbol

$$m_{8,k}(z) = s_{8,k}(z)\; c_k(z)$$

reproducing $EP^8_\Gamma = \mathrm{span}\{\, 1, x, e^{\mathrm{i}tx}, e^{-\mathrm{i}tx}, e^{2\mathrm{i}tx}, e^{-2\mathrm{i}tx}, e^{3\mathrm{i}tx}, e^{-3\mathrm{i}tx} \,\}$.

Towards a closed form expression of $m_{2n+2,k}(z)$

Chebyshev polynomials of the first kind

$$v_k = \cos\left(\frac{t}{2^{k+1}}\right)$$

The degree-j Chebyshev polynomial in v_k is

$$T_j(v_k) = \cos\left(j \arccos(v_k)\right) = \cos\left(\frac{jt}{2^{k+1}}\right)$$

Examples:

$$T_0(v_k) = 1$$
$$T_1(v_k) = v_k$$
$$T_2(v_k) = 2v_k^2 - 1$$
$$T_3(v_k) = 4v_k^3 - 3v_k$$

The exponential B-spline scheme generating EP_Γ^{2n+2}

Proposition 1 [L. R. and A. Viscardi]

$$s_{2n+2,k}(z) = 2\prod_{j=0}^{n} a_{j,k}(z)$$

with $$a_{j,k}(z) = 1 + \frac{(1-z)^2}{2\Big(T_j(v_k)+1\Big)z}, \quad j = 0,\ldots,n$$

☞ Key observation: $\lambda_j(v_k) = 2T_j(v_k)$

Proposition 2

The subdivision scheme with symbols $\{s_{2n+2,k}(z), k \geq 0\}$ is convergent, generates EP_Γ^{2n+2} and has the same regularity as the stationary polynomial B-spline scheme with symbol

$$\frac{(1+z)^{2n+2}}{2^{2n+1}\, z^{n+1}}.$$

The interpolating scheme reproducing EP_Γ^{2n+2}

Theorem 1 [L. R. and A. Viscardi]

$$m_{2n+2,k}(z) = 2a_{0,k}(z) + 2\big(2a_{0,k}(z) - 1\big) \sum_{i=1}^{n} c_i(v_k) \prod_{j=0}^{i-1} \Big(a_{j,k}(-z)\, a_{j,k}(z) \Big)$$

with

$$a_{j,k}(z) = 1 + \frac{(1-z)^2}{2\Big(T_j(v_k)+1\Big)z}, \quad j = 0, \dots, n-1$$

and

$$c_i(v_k) = \frac{2^i}{T_i(v_k)+1} \prod_{\ell=0}^{i-1} \frac{\Big(T_\ell(v_k) - T_{\ell+1}(v_k)\Big)\ \Big(T_\ell(v_k)+1\Big)}{\Big(T_i(v_k) - T_{i+1}(v_k)\Big) - \Big(T_\ell(v_k) - T_{\ell+1}(v_k)\Big)}$$

The closed form derived in Theorem 1 includes the following special cases:

- $n = 1$

C. Beccari, G. Casciola, L. R., A non-stationary uniform tension controlled interpolating 4-point scheme reproducing conics, *CAGD*, 24(1), 1-9, (2007)

- $n = 2$

L. R., From approximating subdivision schemes for exponential splines to high-performance interpolating algorithms, *JCAM*, 224, 383-396, (2009)

- $n = 3$

C. Conti, L. R., Affine combination of B-spline subdivision masks and its non-stationary counterparts, *BIT*, 50, 269-299, (2010)

Proposition 3

The subdivision scheme with symbols $\{m_{2n+2,k}(z), k \geq 0\}$ is interpolatory, convergent, reproduces EP_{Γ}^{2n+2} and has the same regularity as the Dubuc-Deslauriers stationary scheme with symbol

$$\frac{(1+z)^{2n+2}}{2^{2n+1}\, z^{n+1}} \sum_{s=0}^{n} \binom{n+s}{s} (-1)^s \frac{(1-z)^{2s}}{4^s z^s}\,.$$

Benefits of the closed form of $m_{2n+2,k}(z)$

- The computational cost for computing $m_{2n+2,k}(z)$ is remarkably reduced.

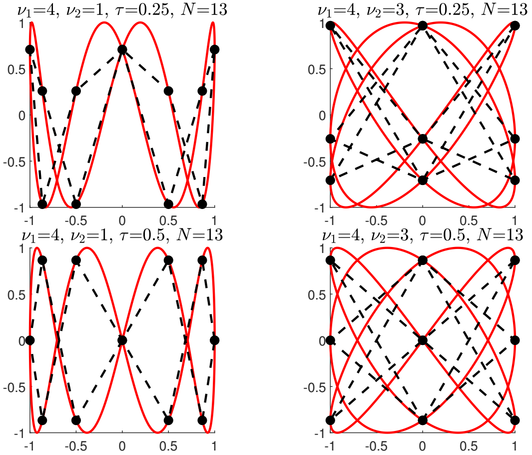
- We can increase n as desired, in order to enlarge the reproduced space and increase the approximation order of the scheme.

C. Conti, L. R., J. Yoon, Approximation order and approximate sum rules in subdivision, *J. Approx. Theory*, 207, 380-401, (2016)

- We can exactly represent Lissajous curves and star-shaped curves with an arbitrary number of convexities, at the desired precision.

In the following we will see application examples of $\{m_{2n+2,k}(z),\ k \geq 0\}$ with $4 \leq n \leq 10$.

Lissajous curves ($n = 4$)



$$x(u) = \cos(\nu_2 u)$$

$$y(u) = \cos\left(\nu_1 u - \frac{\tau\pi}{\nu_2}\right)$$

$$u \in [0, 2\pi]$$

☞ $n = \max\{\nu_1, \nu_2\}$

Figure: Lissajous curves obtained after 6 subdivision steps of the interpolatory scheme $\{m_{10,k}(z),\ k \geq 0\}$ using $t = \sigma = \frac{2\pi}{N-1}$ and starting from the given polygons.

Lissajous curves ($n = 5$)

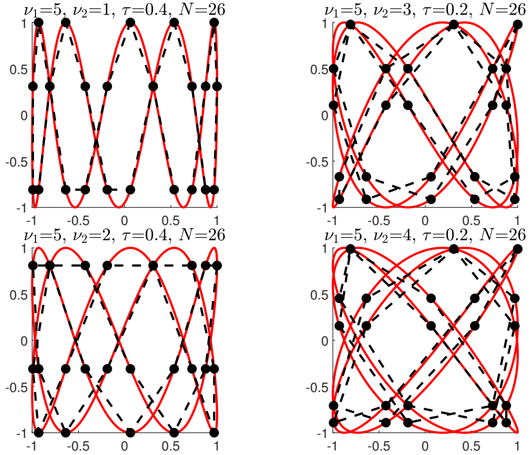


Figure: Lissajous curves obtained after 6 subdivision steps of the interpolatory scheme $\{m_{12,k}(z),\ k \geq 0\}$ using $t = \sigma = \frac{2\pi}{N-1}$ and starting from the given polygons.

$$x(u) = \cos(\nu_2 u)$$
$$y(u) = \cos\left(\nu_1 u - \frac{\tau\pi}{\nu_2}\right)$$
$$u \in [0, 2\pi]$$

☞ $n = \max\{\nu_1, \nu_2\}$

Star-shaped curves

Figure: Examples of real objects with 6-pointed star-shaped profiles.

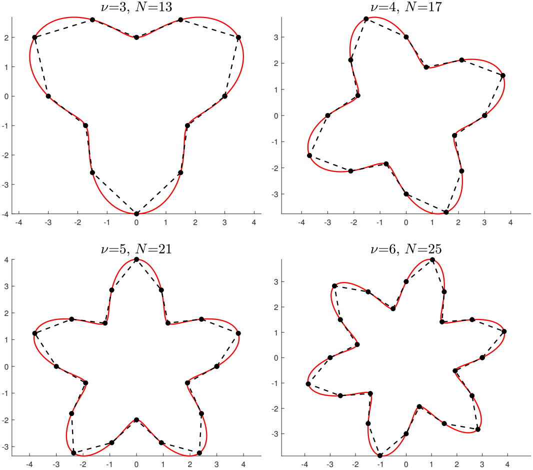


Figure: Planar star-shaped curves obtained after 6 subdivision steps of the interpolatory schemes $\{m_{2n+2,k}(z),\ k \geq 0\}$ with $t = \sigma = \frac{2\pi}{N-1}$, starting from the given polygons.

$$x(u) = \big(3 + \sin(\nu u)\big)\cos(u)$$
$$y(u) = \big(3 + \sin(\nu u)\big)\sin(u)$$
$$u \in [0, 2\pi]$$

with $\nu = 3, 4, 5, 6$

☞ $n = \nu + 1$

$$
\begin{aligned}
x(u) &= \big(3+\sin(\nu u)\big)\cos(u)\\
y(u) &= \big(3+\sin(\nu u)\big)\sin(u)\\
z(u) &= -\frac{1}{4}\big(3+\sin(\nu u)\big)^2
\end{aligned}
$$

$u \in [0, 2\pi]$

with $\nu = 3, 4, 5$

☞ $n = 2\nu$

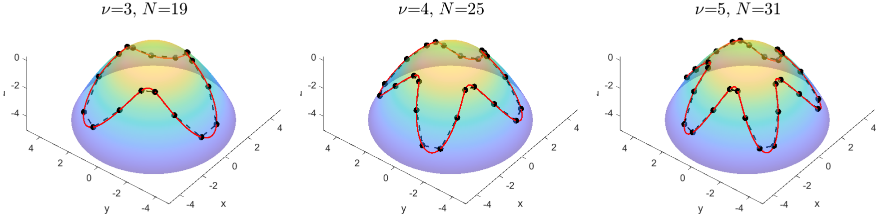


Figure: Spatial star-shaped curves obtained after 6 subdivision steps of the interpolatory schemes $\{m_{2n+2,k}(z),\ k \geq 0\}$ with $t = \sigma = \frac{2\pi}{N-1}$, starting from the given polygons.

Conclusions

Summary of the main contributions of our work:

- having discovered a link between level-dependent subdivision schemes and Chebyshev polynomials;
- having found a closed form expression for the subdivision symbols of exponential B-splines generating integer powers of exponentials;
- having found a closed form expression for the subdivision symbols of the interpolating schemes that reproduce integer powers of exponentials and are asymptotically similar to Dubuc-Deslauriers stationary schemes.


danke
謝謝
ngiyabonga
teşekkür ederim
спасибо
thank you
gracias
dank je
bedankt
hvala
maururu
tapadh leat
mochchakkeram
dziękuję
go raibh maith agat
obrigado
sukriya
kop khun krap
arigato
dakujem
grazie
terima kasih
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merci

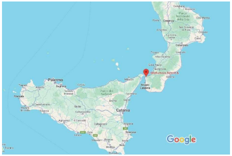
Announcement

SMART 2025

4th international conference on

Subdivision, Geometric and Algebraic Methods, Isogeometric Analysis and Refinability in ITaly

Santa Trada di Cannitello (Reggio Calabria), Sep 28 – Oct 2, 2025



SAVE THE DATE !!!

Confirmed invited speakers: Keenan Crane, Annie Cuyt, Stefano De Marchi, Carlotta Giannelli, Marjeta Knez, Deepesh Toshniwal, Jungho Yoon

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