Random Leja points for interpolation

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SIGMA 2024 Workshop, CIRM, Marseille, October 28th, 2024

Context:

- $K \subset \mathbb{C}$ a compact set
- $n + 1$ distinct points $z_0, \ldots, z_n$ in K

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Polynomial interpolation: for $f \in C^0(K, \mathbb{C})$ known at the z_i's, approximate f by its unique Lagrange interpolating polynomial $L_n(f) \in \mathbb{C}_n[X]$ satisfying

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Goal: choose points $z_0, \dots, z_n$ ensuring good approximation properties as $n \to +\infty$.

What does one mean by "good"?

Define $\|g\|_K := \sup_{z\in K} |g(z)|$ for $g \in C^0(K, \mathbb{C})$.

- Minimal expectation: a family of points $n \mapsto z_0, \ldots, z_n$ is said to be *extremal* if

$$\text{for all } f \text{ holomorphic in a neighbourhood of } K, \quad \|L_n(f) - f\|_K \longrightarrow 0.$$

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- More ambitious: associated Lebesgue constant Λ_n is of moderate growth

$$\Lambda_n := \sup_{z\in K} \sum_{i=0}^{n} \prod_{j\neq i} \left| \frac{z - z_j}{z_i - z_j} \right|$$

  quantifies approximation quality from the basic estimate

$$\|L_n(f) - f\|_K \leq (1 + \Lambda_n) \inf_{P\in\mathbb{C}_n[X]} \|P - f\|_K.$$

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Finding good points is notoriously hard:

- equidistant points typically are not extremal...
- Polynomially-growing Lebesgue constants is quite satisfactory,
- logarithmic growth is the Holy Grail.

Leja points

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- ⋄ Lebesgue constants
  - for generic K, subexponential Lebesgue constant $\Delta_n^{1/n} \to 1$ (Totik '23)
  - for $K = D(0,1)$, $\Delta_n = O(n)$ (Chkifa '13),
  - for $K = [-1,1]$, $\Delta_n = O(n)$ is conjectured, best known estimate is $\Delta_n = O(n^{13/4})$ (Andrievskii and Nazarov '22)...

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Problem: how can one compute them efficiently? More difficult than it may seem.

Be careful with Leja points

Usual implementation: fix a grid $\mathcal{A}$ once and for all and maximise over the grid, i.e.,

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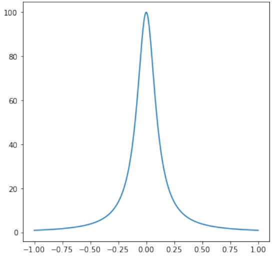

Figure: Function to be interpolated, i.e., $z \mapsto \frac{1}{z^2+0.1^2}$.

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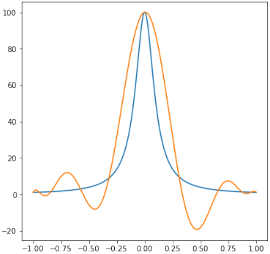

Figure: Function to be interpolated, i.e., $z \mapsto \frac{1}{z^2+0.1^2}$, grid of equidistant points of size 500, Lagrange interpolating polynomial $L_n(f)$ for $n = 10$.

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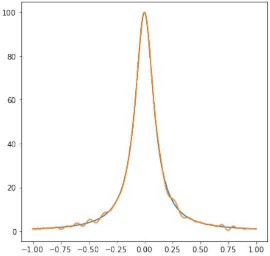

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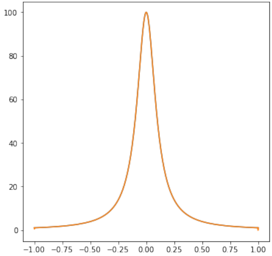

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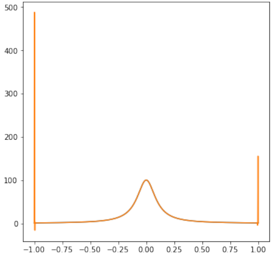

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Pseudo-Leja points

Leja points: sequence $(z_n) \in K^{\mathbb{N}}$ such that

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Pseudo-Leja points (Białas-Cież and Calvi '12): sequence $(z_n) \in K^{\mathbb{N}}$ such that

$$|\pi_n(z_n)| \geq \tau_n \|\pi_n\|_K, \qquad \pi_n(z) := \prod_{i=0}^{n-1}(z - z_i)$$

where $0 < \tau_n \leq 1$ is subexponential, i.e., satisfies $\tau_n^{1/n} \to 1$ as $n \to +\infty$.
If $\tau_n \sim n^{-\beta}$, pseudo-Leja points of order β.

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If $\tau_n \sim n^{-\beta}$, pseudo-Leja points of order β.

Known results:

- for generic K, pseudo-Leja points are still extremal
- Lebesgue constants? Not much except for pseudo-Leja points of order $\beta = 0$.

Several ways to build so-called weakly admissible meshes (Calvi - Levenberg '08)

Idea: choose $z_n \in \mathcal{A}_n$ such that, recalling $\pi_n(z) = \prod_{i=0}^{n-1}(z - z_i)$,

$$|\pi_n(z_n)| = \max_{z \in \mathcal{A}_n} |\pi_n(z)|,$$

where $\mathcal{A}_n$ is discrete finite set of cardinal N_n.

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Different recipes to compute such meshes that lead to pseudo-Leja points (Białas-Cież and Calvi '12):

- For sets with C^1 boundaries, $N_n \sim n^{r_m}$ where r_m (generically $r_m \in \{1, 2\}$) stems from Markov's inequality

$$\sup_{P \in \mathbb{C}_n[X] \setminus \{0\}} \frac{\|P'\|_K}{\|P\|_K} \lesssim n^{r_m}.$$

  Requires to parameterise the boundary.
- $N_n \sim n$ for polygons at the price of parameterising each edge.

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Main issue: lack of modularity.

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All results can be found in my preprint: *Random Leja points*, arXiv:2406.11499

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Lemma: let (Z_n) be any sequence of random variables such that for some $\beta \geq 0$,

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Proof. Borel-Cantelli's Lemma implies that a.s. $|\pi_n(Z_n)| \geq n^{-\beta} \|\pi_n\|_K$ for n large enough.
$\implies$ the points Z_n a.s. are pseudo Leja points of order β, hence a.s. extremal. □

Random Leja points are extremal

Recall $\pi_n(z) = \prod_{i=0}^{n-1}(z - Z_i)$, Z_n of density $\frac{|\pi_n|}{\|\pi_n\|_1}$ conditionally on $\mathcal{F}_n := \sigma(Z_0, \dots Z_{n-1})$.

Theorem

For generic K and σ, random Leja points are almost surely extremal.

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Proof. By Markov's inequality + definition of random Leja points

$$\mathbb{P}\left(|\pi_n(Z_n)| < n^{-\beta}\|\pi_n\|_\infty \,\middle|\, \mathcal{F}_n\right) \leq n^{-\beta}\|\pi_n\|_\infty \,\mathbb{E}\left(|\pi_n(Z_n)|^{-1} \,\middle|\, \mathcal{F}_n\right)$$

$$= n^{-\beta}\|\pi_n\|_\infty \frac{1}{\|\pi_n\|_1}\int_K |\pi_n(z)|^{-1}|\pi_n(z)|\, d\sigma(z) = n^{-\beta}\sigma(K)\frac{\|\pi_n\|_\infty}{\|\pi_n\|_1}$$

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Nikolskii inequality for any reasonable K and σ:

$$\sup_{P\in\mathbb{C}_n[X]\setminus\{0\}} \frac{\|P\|_\infty}{\|P\|_1} \lesssim n^{r_\ell}.$$

Leads to convergence of $\sum \mathbb{P}\left(|\pi_n(Z_n)| < n^{-\beta}\|\pi_n\|_K\right)$ for any $\beta > r_\ell + 1$. □

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Caveat: as "theoretical" as Leja points already were!

Metropolis-Hastings random Leja points

$Z_0, \ldots, Z_{n-1}$ known, want to draw Z_n of density $\frac{|\pi_n|}{\|\pi_n\|_1}$ while only sampling from the uniform distribution $\mathcal{U}_\sigma(K)$

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- ◇ in practice, halted at the kth iterate...

In other words, draw $(X_k)_{k\in\mathbb{N}}$ i.i.d $\mathcal{U}_\sigma(K)$, $(U_k)_{k\in\mathbb{N}}$ i.i.d $\mathcal{U}([0,1])$.

$$\begin{cases} Y_0 = X_0 \\ Y_k = \begin{cases} X_k & \text{if } U_k \leq \min\left(1, \frac{|\pi_n(X_k)|}{|\pi_n(Y_{k-1})|}\right) \\ Y_{k-1} & \text{else} \end{cases} \end{cases}$$

Set $Z_n := Y_k$ for appropriately chosen $k = N_n$: *Metropolis Hastings (MH)* random Leja points.

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How to choose N_n? Convergence results for various distances, in the form

$$d(\mu, \mu_k) \lesssim \left(1 - \frac{1}{M}\right)^k,$$

where M is a bound such that

$$\frac{|\pi_n|}{\|\pi_n\|_1} \leq M \frac{1}{\sigma(K)} \quad a.e.$$

Extremality of MH random Leja points (1)

Theorem

For generic K and σ, choose

$$N_n \sim n^{\alpha}, \quad \textit{with} \quad \alpha > r_\ell.$$

Then MH random Leja points are almost surely extremal.

Generically, r_ℓ from Nikolskii's inequality $\sup_{0 \le \deg(P) \le n} \frac{\|P\|_\infty}{\|P\|_1} \lesssim n^{r_\ell}$ satisfies $r_\ell \in \{1, 2\}$.

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$$\mathbb{P}\Big(|\pi_n(Z_n)| < n^{-\beta}\|\pi_n\|_\infty \,\Big|\, \mathcal{F}_n\Big)$$
$$\leq \mathbb{P}\Big(|\pi_n(\tilde{Z}_n)| < 2n^{-\beta}\|\pi_n\|_\infty \,\Big|\, \mathcal{F}_n\Big) + \mathbb{P}\left(\Big||\pi_n(Z_n)| - |\pi_n(\tilde{Z}_n)|\Big| > n^{-\beta}\|\pi_n\|_\infty \,\Big|\, \mathcal{F}_n\right)$$

First term already dealt with:

$$\mathbb{P}\left(|\pi_n(\tilde{Z}_n)| < 2n^{-\beta}\|\pi_n\|_\infty \,\Big|\, \mathcal{F}_n\right) \lesssim n^{-\beta} n^{r_\ell}.$$

Extremality of MH random Leja points (2)

For the second term, use Markov's inequality as well as Markov's inequality (!)

$$\mathbb{P}\Big(\Big| |\pi_n(Z_n)| - |\pi_n(\tilde{Z}_n)| \Big| > n^{-\beta} \|\pi_n\|_\infty \,\Big|\, \mathcal{F}_n \Big)$$
$$\leq \mathbb{P}\left[e^{c_m n^{r_m} |Z_n - \tilde{Z}_n|} - 1 > n^{-\beta} \,\Big|\mathcal{F}_n \right] \lesssim n^{r_m} n^{\beta}\, \mathbb{E}\left[\left| Z_n - \tilde{Z}_n \right| \Big| \mathcal{F}_n \right]$$

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Choose $\tilde{Z}_n$ to be coupled to Z_n in such a way that (conditionally on $\mathcal{F}_n$), it realises the infimum within the 1-Wasserstein distance, that is,

$$\mathbb{E}\left[\big|Z_n - \tilde{Z}_n\big| \Big| \mathcal{F}_n \right] = W(\mu_{Z_n}, \mu_{\tilde{Z}_n}) \lesssim \left(1 - \frac{1}{M_n}\right)^{N_n},$$

where M_n is any constant such that $\frac{\|\pi_n\|_\infty}{\|\pi_n\|_1} \leq M_n \frac{1}{\sigma(K)}$: can take $M_n \sim n^{r_\ell}$.

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Conclusion: estimate

$$\mathbb{P}\left(|\pi_n(Z_n)| < n^{-\beta} \|\pi_n\|_\infty \right) \lesssim n^{-\beta} n^{r_\ell} + n^{r_m} n^{\beta} \left(1 - \frac{1}{M_n}\right)^{N_n},$$

$\implies$ convergence of $\sum \mathbb{P}\left(|\pi_n(Z_n)| < n^{-\beta} \|\pi_n\|_K \right)$ for any $\beta > r_\ell + 1$ since $N_n \sim n^{\alpha}$, $\alpha > r_\ell$. □

Randomised weakly admissible meshes

Randomised weakly admissible meshes proposed in (Xu and Narayan '23) lead to

Randomised mesh (RM) random Leja points:

$$Z_n \in \underset{1 \le k \le N_n}{\arg\max} |\pi_n(X_k)|.$$

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Using techniques of proofs inspired by Xu and Narayan '23 and the recent result of Totik '23

Theorem

For generic K and σ, choose

$$N_n \sim n^\alpha, \quad \textit{with} \quad \alpha > r_m\, r_c.$$

Then RM random Leja points almost surely have subexponential Lebesgue constant.

Generically, $r_m \in \{1, 2\}$ and $r_c \in \{1, 2\}$ (associated to covering numbers for K, σ)

Disk

Toy example (equidistant points on the boundary are already excellent).
MH and RM random Leja points; here $r_\ell = 2$, $r_m r_c = 1 \times 2$.

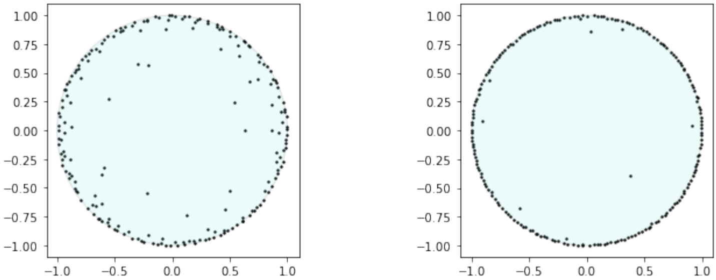

Figure: Example of $n = 200$ MH points (left figure) and RM points (right figure), with $N_n = \lfloor n^{2+\varepsilon} \rfloor$ in both cases, for $\varepsilon = 0.01$.

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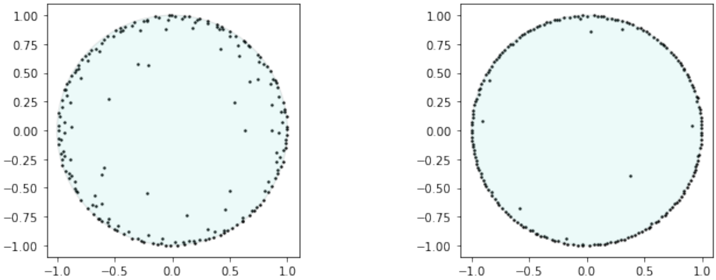
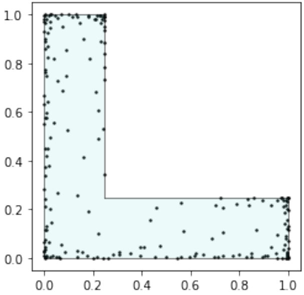


Figure: Example of $n = 200$ MH points (left figure) and RM points (right figure), with $N_n = \lfloor n^{2+\varepsilon} \rfloor$ in both cases, for $\varepsilon = 0.01$.

	MH points	RM points
$\mathbb{E}[\Lambda_n]$	2.92	0.50
$\sqrt{\mathrm{Var}(\Lambda_n)}$	2.99	0.51

Table: Estimates for polynomial growth of $\mathbb{E}[\Lambda_n]$ and $\sqrt{\mathrm{Var}(\Lambda_n)}$.

MH random Leja points for polygons. RM Leja points are intractable, due to $r_m r_c = 2 \times 2$.



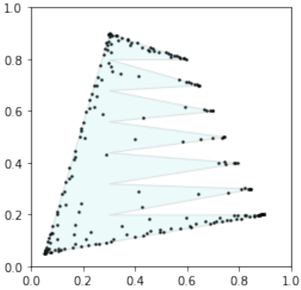


Figure: Example of $n = 200$ MH points for two polygons, with $N_n = \lfloor n^{2+\varepsilon} \rfloor$, $\varepsilon = 0.01$.

Summary

	MH points	RM points	pseudo-Leja points
modularity	✓	✓	✗
reproducibility	✗	✗	✓
order as pseudo-Leja points (accuracy)	$\sim 1 + r_\ell$	0	0
number of underlying points (complexity)	r_ℓ	$r_m r_c$	r_m

Table: Comparison between different methods.

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number of underlying points (complexity)	r_ℓ	$r_m r_c$	r_m

Table: Comparison between different methods.

And some open questions:

- Higher dimension? curse of dimensionality because r_ℓ, r_c scale linearly with dimension... r_m, however, does not.
- Estimates for Lebesgue constants, at least on average?
- Alternative clever sampling strategies?