MOCCA: A Fast Algorithm for Parallel MRI Reconstruction Using Model Based Coil Calibration

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Outline

- The parallel Magnetic Resonance Imaging (MRI) model
- Overview of reconstruction methods
- Modelling of coil sensitivities
- Reconstruction from incomplete *k*-space measurements
- Numerical examples
- Summary

Joint work with Yannick Riebe

DFG project in the CRC 1456 together with Martin Uecker (TU Graz)

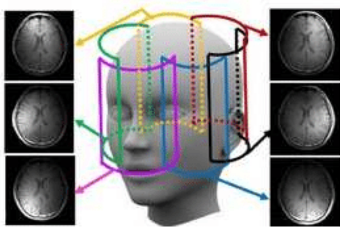
Reconstruction problem in parallel MRI

Given: (incomplete) discrete measurements for N_c receiver channels

$$y_{\boldsymbol{\nu}}^{(j)} := y^{(j)}\left(\frac{\boldsymbol{\nu}}{N}\right) = \int_{\Omega} s^{(j)}(\boldsymbol{x}) m(\boldsymbol{x})\, \mathrm{e}^{-2\pi\mathrm{i}\frac{\boldsymbol{\nu}}{N}\cdot\boldsymbol{x}} d\boldsymbol{x} + n^{(j)}\left(\frac{\boldsymbol{\nu}}{N}\right),$$

for $\boldsymbol{\nu} \in \Lambda_N := \{-\frac{N}{2}, \ldots, \frac{N}{2} - 1\} \times \{-\frac{N}{2}, \ldots, \frac{N}{2} - 1\}, \quad j = 0, \ldots, N_c - 1.$

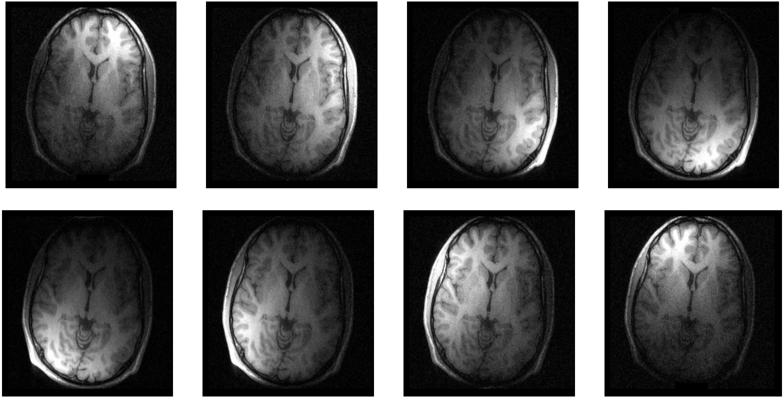
m	magnetization image (complex)
$s^{(j)}$	complex valued sensitivity profiles of the N_c individual coils
$n^{(j)}$	noise term
Ω	bounded area of interest, here $\Omega = [-\frac{N}{2}, \frac{N}{2}]^2$.



(Mardani et al. (2016))

Wanted: m and $s^{(j)}$, $j = 0, \ldots, N_c - 1$.

Fully sampled coil images after inverse Fourier transform



Test data of the data set brain_8ch in the ESPIRiT toolbox.

Overview of reconstruction methods (incomplete)

- **Given sensitivities:**
  SENSE (Pruessmann et al. '99))
- **Approximation of unacquired data using calibration data:**
  SMASH (Jakob et al. '98, Heidemann et al. '01),
  GRAPPA (Griswold et al. '02),
  SPIRiT (Lustig et al. '10)
- **Subspace methods using calibration data:**
  ESPIRiT (Uecker et al. '14)
  PISCO (Lobos et al '24)
- **Low rank matrix completion (no calibration):**
  SAKE (Shin et al. '14),
  ALOHA (Jin et al. '15),
  LORAKS (Haldar et al. '14)
- **Nonlinear optimization problem (no calibration):**
  NLINV (Uecker et al. '08), Keeling et al '12, Allison et al. '13,
  BARISTA (Muckley et al. '15),
  ENLIVE (Holme et al. '19)

Discrete model

Discrete model: $\boldsymbol{y}^{(j)} = (y_{\boldsymbol{\nu}}^{(j)})_{\boldsymbol{\nu}\in\Lambda_N} = \mathcal{F}\,(\boldsymbol{m} \circ \mathbf{s}^{(j)}) \qquad j = 0, \ldots, N_c - 1$

where $\mathcal{F} = \mathcal{F}_{N^2} := (\omega_N^{\boldsymbol{\nu}\cdot\boldsymbol{n}})_{\boldsymbol{\nu},\boldsymbol{n}\in\Lambda_N}$, $\boldsymbol{m} := (m_{\boldsymbol{n}})_{\boldsymbol{n}\in\Lambda_N}$, $\mathbf{s}^{(j)} := (s^{(j)}(\boldsymbol{n}))_{\boldsymbol{n}\in\Lambda_N}$.

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where $\mathcal{F} = \mathcal{F}_{N^2} := (\omega_N^{\boldsymbol{\nu}\cdot\boldsymbol{n}})_{\boldsymbol{\nu},\boldsymbol{n} \in \Lambda_N}$, $\boldsymbol{m} := (m_{\boldsymbol{n}})_{\boldsymbol{n} \in \Lambda_N}$, $\mathbf{s}^{(j)} := (s^{(j)}(\boldsymbol{n}))_{\boldsymbol{n} \in \Lambda_N}$.

Problem

Reconstruct $\boldsymbol{m} := (m_{\boldsymbol{n}})_{\boldsymbol{n} \in \Lambda_N}$ and $\mathbf{s}^{(j)} := (s^{(j)}(\boldsymbol{n}))_{\boldsymbol{n} \in \Lambda_N}$ from undersampled data

$$\mathcal{P}\boldsymbol{y}^{(j)} = \mathcal{P}\mathcal{F}\,(\boldsymbol{m} \circ \mathbf{s}^{(j)}) \qquad j = 0, \ldots, N_c - 1$$

where $\mathcal{P}$ chooses only the acquired locations in k-space.

MOCCA (Model based coil calibration) [Plonka & Riebe '24]

Given incomplete measurements:

$$\mathcal{P}\boldsymbol{y}^{(j)} = \mathcal{P}\,\mathcal{F}\,(\boldsymbol{m} \circ \mathbf{s}^{(j)}), \quad j = 0, \ldots, N_c - 1$$

Auto calibration signal (ACS) region $\Lambda_M \subset \Lambda_{\mathcal{P}} \subset \Lambda_N$

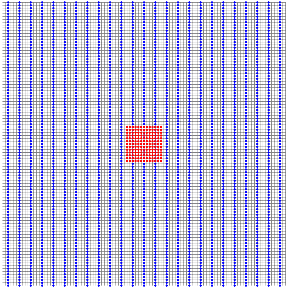

Illustration of the grid Λ_N ($N = 100$), the ACS region and the grid of incomplete measurements outside (here each fourth column)

MOCCA: Modelling sensitivities as trigonometric polynomials

Let $\Lambda_L := \{-n, \ldots, n\} \times \{-n, \ldots, n\}$, $L = 2n + 1 \ll N$.

Model for $\mathbf{s}^{(j)} = (s_{\boldsymbol{n}}^{(j)})_{\boldsymbol{n} \in \Lambda_N}$:

$$s_{\boldsymbol{n}}^{(j)} := \sum_{\boldsymbol{r} \in \Lambda_L} c_{\boldsymbol{r}}^{(j)} \, \mathrm{e}^{\frac{2\pi \mathrm{i}}{N} \boldsymbol{n} \cdot \boldsymbol{r}}, \quad j = 0, \ldots, N_c - 1,$$

where $\boldsymbol{n} \cdot \boldsymbol{r} = k_1 r_1 + k_2 r_2$, $c_{\boldsymbol{r}}^{(j)} \in \mathbb{C}$.

Matrix form: $\mathbf{s}^{(j)} := (s_{\boldsymbol{n}}^{(j)})_{\boldsymbol{n} \in \Lambda_N} = \overline{\mathcal{F}}_{N,L} \, \boldsymbol{c}^{(j)}, \quad j = 0, \ldots, N_c - 1,$

where $\mathcal{F}_{N,L} := (\omega_N^{\boldsymbol{r} \cdot \boldsymbol{n}})_{\boldsymbol{n} \in \Lambda_N, \boldsymbol{r} \in \Lambda_L}$ and $\mathbf{c}^{(j)} := (c_{\boldsymbol{r}}^{(j)})_{\boldsymbol{r} \in \Lambda_L} \in \mathbb{C}^{L^2}$

Generalized model and ambiguities

The coil sensitivity model can be generalized to

$$\tilde{s}_{\boldsymbol{n}}^{(j)} := \gamma_{\boldsymbol{n}} \, s_{\boldsymbol{n}}^{(j)} = \gamma_{\boldsymbol{n}} \sum_{\boldsymbol{r} \in \Lambda_L} c_{\boldsymbol{r}}^{(j)} \, \omega_N^{-\boldsymbol{r}\cdot\boldsymbol{n}} \qquad j = 0, \ldots, N_c - 1, \ \boldsymbol{n} \in \Lambda_N,$$

where $\gamma_{\boldsymbol{n}} \neq 0$ does not depend on j. Then,

$$\boldsymbol{y}^{(j)} = \mathcal{F}(\boldsymbol{m} \circ \mathbf{s}^{(j)}) = \mathcal{F}(\underbrace{\boldsymbol{m} \circ \boldsymbol{\gamma}^{-1}}_{\tilde{\boldsymbol{m}}} \circ \underbrace{\boldsymbol{\gamma} \circ \mathbf{s}^{(j)}}_{\tilde{\mathbf{s}}^{(j)}}) = \mathcal{F}(\tilde{\boldsymbol{m}} \circ \tilde{\mathbf{s}}^{(j)})$$

with $\boldsymbol{\gamma} := (\gamma_{\boldsymbol{n}})_{\boldsymbol{n} \in \Lambda_N}$ $\boldsymbol{\gamma}^{-1} := (\gamma_{\boldsymbol{n}}^{-1})_{\boldsymbol{n} \in \Lambda_N}$,

solution $(\boldsymbol{m}, (\mathbf{s}^{(j)})_{j=0}^{N_c-1}) \quad \Longrightarrow \quad$ many solutions $(\tilde{\boldsymbol{m}}, (\tilde{\mathbf{s}}^{(j)})_{j=0}^{N_c-1})$

Ambiguities and the sum-of-squares condition

Many recovery algorithms approximate the unacquired data $y_{\boldsymbol{n}}^{(j)}$ from the acquired data in a first step and apply

$$m_{\boldsymbol{n}} = \Big(\sum_{j=0}^{N_c-1} |\check{y}_{\boldsymbol{n}}^{(j)}|^2\Big)^{\frac{1}{2}}, \qquad \boldsymbol{n} \in \Lambda_N, \qquad (\text{ground truth})$$

Taking $\tilde{s}_{\boldsymbol{n}}^{(j)} := \gamma_{\boldsymbol{n}}\, s_{\boldsymbol{n}}^{(j)}$ and $\tilde{m}_{\boldsymbol{n}} = \frac{1}{\gamma_{\boldsymbol{n}}} m_{\boldsymbol{n}}$ with with

$$\gamma_{\boldsymbol{n}} := \begin{cases} \operatorname{sign}(m_{\boldsymbol{n}})\Big(\sum_{j=0}^{N_c-1} |s_{\boldsymbol{n}}^{(j)}|^2\Big)^{-1/2}, & \boldsymbol{n} \in \Lambda_N,\ \sum_{j=0}^{N_c-1} |s_{\boldsymbol{n}}^{(j)}|^2 > 0, \\ 0 & \text{otherwise,} \end{cases}$$

Then

$$\sum_{j=0}^{N_c-1} |\tilde{s}_{\boldsymbol{n}}^{(j)}|^2 = 1 \quad \text{for } \gamma_{\boldsymbol{n}} \neq 0 \quad \text{and} \quad \tilde{m}_{\boldsymbol{n}} = \Big(\sum_{j=0}^{N_c-1} |\check{y}_{\boldsymbol{n}}^{(j)}|^2\Big)^{\frac{1}{2}}.$$

MOCCA (Model based coil calibration)

Given: $\boldsymbol{y}^{(j)} = (y_{\boldsymbol{\nu}}^{(j)})_{\boldsymbol{\nu} \in \Lambda_N} = \mathcal{F}\,(\boldsymbol{m} \circ \mathbf{s}^{(j)}), \qquad j = 0, \ldots, N_c - 1.$

Step 1: Reconstruction of $\mathbf{s}^{(j)} = \overline{\mathcal{F}}_{N,L}\, \boldsymbol{c}^{(j)}$

We have $\check{\mathbf{y}}^{(j)} := \mathcal{F}^{-1}\mathbf{y}^{(j)} = \boldsymbol{m} \circ \mathbf{s}^{(j)} = \boldsymbol{m} \circ (\overline{\mathcal{F}}_{N,L}\mathbf{c}^{(j)})$

and

$$\check{\boldsymbol{y}}^{(j)} \circ \mathbf{s}^{(j')} = \mathbf{s}^{(j)} \circ \mathbf{s}^{(j')} \circ \boldsymbol{m} = \mathbf{s}^{(j)} \circ \check{\boldsymbol{y}}^{(j')}$$

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and

$$\check{\boldsymbol{y}}^{(j)}\circ\mathbf{s}^{(j')} = \mathbf{s}^{(j)}\circ\mathbf{s}^{(j')}\circ\boldsymbol{m} = \mathbf{s}^{(j)}\circ\check{\boldsymbol{y}}^{(j')}$$

Then

$$(\check{\boldsymbol{y}}^{(j)})\circ(\sum_{j'\neq j}\mathbf{s}^{(j')}) = (\sum_{j'\neq j}\check{\boldsymbol{y}}^{(j')})\circ\mathbf{s}^{(j)}$$

i.e.,

$$(\check{\boldsymbol{y}}^{(j)}\circ\overline{\mathcal{F}}_{N,L}\sum_{j'\neq j}\boldsymbol{c}^{(j')}) - (\sum_{j'\neq j}\check{\boldsymbol{y}}^{(j')}\circ\overline{\mathcal{F}}_{N,L}\,\boldsymbol{c}^{(j)}) = \mathbf{0}.$$

MOCCA: Reconstruction of $\mathbf{s}^{(j)}$ from the calibration area

Multiplying this equation with the $2D$-Fourier matrix $\mathcal{F}$ we obtain

$$[-\mathcal{F}\,\mathrm{diag}(\sum_{j'\neq j}\check{\boldsymbol{y}}^{(j')})\overline{\mathcal{F}}_{N,L},\ \mathcal{F}\,\mathrm{diag}(\check{\boldsymbol{y}}^{(j)})\,\overline{\mathcal{F}}_{N,L}]\begin{pmatrix}\boldsymbol{c}^{(j)}\\ \sum\limits_{j'\neq j}\boldsymbol{c}^{(j')}\end{pmatrix}=\mathbf{0}.$$

Hence, with $\mathbf{Y}_{N,L}^{(j)} := (y^{(j)}_{(\boldsymbol{\nu}-\boldsymbol{r}) \bmod \Lambda_N})_{\boldsymbol{\nu}\in\Lambda_N,\boldsymbol{r}\in\Lambda_L} \in \mathbb{C}^{N^2\times L^2}$,

$$\begin{bmatrix} -(\sum\limits_{\ell\neq 0}\mathbf{Y}_{N,L}^{(\ell)}) & \mathbf{Y}_{N,L}^{(0)} & \mathbf{Y}_{N,L}^{(0)} & \dots & \mathbf{Y}_{N,L}^{(0)} \\ \mathbf{Y}_{N,L}^{(1)} & -(\sum\limits_{\ell\neq 1}\mathbf{Y}_{N,L}^{(\ell)}) & \mathbf{Y}_{N,L}^{(1)} & \dots & \mathbf{Y}_{N,L}^{(1)} \\ \mathbf{Y}_{N,L}^{(2)} & \mathbf{Y}_{N,L}^{(2)} & -(\sum\limits_{\ell\neq 2}\mathbf{Y}_{N,L}^{(\ell)}) & \dots & \mathbf{Y}_{N,L}^{(2)} \\ \vdots & & & \ddots & \vdots \\ \mathbf{Y}_{N,L}^{(N_c-1)} & \dots & \mathbf{Y}_{N,L}^{(N_c-1)} & \mathbf{Y}_{N,L}^{(N_c-1)} & -(\sum\limits_{\ell\neq N_c-1}\mathbf{Y}_{N,L}^{(\ell)}) \end{bmatrix}\begin{bmatrix}\mathbf{c}^{(0)}\\ \mathbf{c}^{(1)}\\ \mathbf{c}^{(2)}\\ \vdots\\ \mathbf{c}^{(N_c-1)}\end{bmatrix}=\mathbf{0},$$

i.e., reducing from N to M

$$\mathbf{A}_{M,L}\mathbf{c}=\mathbf{0} \qquad \text{with} \qquad \mathbf{A}_{M,L}\in\mathbb{C}^{N_cM^2\times N_cL^2}.$$

MOCCA: Reconstruction from incomplete measurements

Step 1: Reconstruction of $\mathbf{s}^{(j)}$ from data in the calibration region:

- Build $\mathbf{A}_{M,L}$ from the *k*-space data in the calibration region, compute $\boldsymbol{c}^{(j)}$ by solving $\mathbf{A}_{M,L}\mathbf{c} = \mathbf{0}$.
- Compute $\mathbf{s}^{(j)} = \overline{\mathcal{F}}_{N,L}\, \boldsymbol{c}^{(j)}$.

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Normalize the sensitivities:

- Compute $\mathbf{d} = (d_{\boldsymbol{n}})_{\boldsymbol{n}\in\Lambda_N} := \sum_{j=0}^{N_c-1} \overline{\mathbf{s}^{(j)}} \circ \mathbf{s}^{(j)}$.
- Define $\tilde{\mathbf{s}}^{(j)} := (\mathbf{d}^{+})^{\frac{1}{2}} \circ \mathbf{s}^{(j)}$, such that $\sum_{j=0}^{N_c-1} |\tilde{s}_{\boldsymbol{n}}^{(j)}|^2 = 1$, $\boldsymbol{n} \in \Lambda_N$.

Then,

$$\mathbf{y}^{(j)} = \mathcal{F}(\mathbf{s}^{(j)} \circ \mathbf{m}) = \mathcal{F}(\tilde{\mathbf{s}}^{(j)} \circ \tilde{\mathbf{m}}) \qquad \text{with} \qquad \tilde{\boldsymbol{m}} := \mathbf{d}^{\frac{1}{2}} \circ \boldsymbol{m}.$$

MOCCA: Reconstruction from incomplete measurements

Step 2: Reconstruction of $\boldsymbol{m}$.

Solve

$$\tilde{\boldsymbol{m}} := \operatorname*{argmin}_{\boldsymbol{m} \in \mathbb{C}^{N^2}} \left(\sum_{j=0}^{N_c-1} \| \mathcal{P} \boldsymbol{y}^{(j)} - (\mathcal{P}\, \mathcal{F}\, (\tilde{\mathbf{s}}^{(j)} \circ \boldsymbol{m}) \|_2^2 \right).$$

We obtain the linear system

$$\Big(\sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \, \mathbf{G}^{(j)} \Big) \tilde{\boldsymbol{m}} = \sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \, \mathcal{P} \mathbf{y}^{(j)}.$$

with $\mathbf{G}^{(j)} := \mathcal{P}\mathcal{F}\mathrm{diag}(\tilde{\mathbf{s}}^{(j)})$.

Step 2: Reconstruction of $\boldsymbol{m}$.

Solve

$$\tilde{\boldsymbol{m}} := \operatorname*{argmin}_{\boldsymbol{m}\in\mathbb{C}^{N^2}} \left(\sum_{j=0}^{N_c-1} \| \mathcal{P} \boldsymbol{y}^{(j)} - (\mathcal{P}\, \mathcal{F}\, (\tilde{\mathbf{s}}^{(j)} \circ \boldsymbol{m}) \|_2^2 \right).$$

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with $\mathbf{G}^{(j)} := \mathcal{P}\mathcal{F}\mathrm{diag}(\tilde{\mathbf{s}}^{(j)})$.

Regularization:

$$\Big(\beta \mathbf{I} + \sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \, \mathbf{G}^{(j)} \Big) \tilde{\boldsymbol{m}} = \sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \, \mathcal{P} \mathbf{y}^{(j)}$$

with $\beta > 0$. However, for $\beta > 0$ an exact reconstruction is not longer obtained, even if the original matrix $\sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \, \mathbf{G}^{(j)}$ is positive definite.

Algorithm for reconstruction from incomplete measurements

Input: $\mathcal{P}\boldsymbol{y}^{(j)}$ $j = 0, \ldots, N_c - 1$.
$L \leq M \ll N$ (size of index sets)

1. Build $\mathbf{A}_{M,L}$ from calibration data and solve $\mathbf{A}_{M,L}\mathbf{c} = \mathbf{0}$. Extract $\mathbf{c}^{(j)}$.
2. Compute $\mathbf{s}^{(j)} = \overline{\mathcal{F}}_{N,L}\boldsymbol{c}^{(j)}$, $j = 0, \ldots, N_c - 1$, and $\mathbf{d}^{+}$.
3. Normalization: For $j = 0 : N_c - 1$ compute $\tilde{\mathbf{s}}^{(j)} := (\mathbf{d}^{+})^{\frac{1}{2}} \circ \mathbf{s}^{(j)}$.
4. Solve the equation system

$$\left(\beta\,\mathbf{I} + \sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \mathbf{G}^{(j)}\right) \tilde{\boldsymbol{m}} = \sum_{j=0}^{N_c-1} (\mathbf{G}^{(j)})^* \,\mathcal{P}\, \boldsymbol{y}^{(j)}$$

   with $\mathbf{G}^{(j)} := \mathcal{P}\mathcal{F}\,\tilde{\mathbf{s}}^{(j)}$ and $\beta \geq 0$.

5. Compute $\tilde{\boldsymbol{m}} = \mathrm{sign}(\boldsymbol{m}) \circ \boldsymbol{m}$ and $\tilde{\mathbf{s}}^{(j)} = \mathrm{sign}(\boldsymbol{m}) \circ \tilde{\mathbf{s}}^{(j)}$, $j = 0, \ldots, N_c - 1$.

Output: $\tilde{\boldsymbol{m}}$, $\tilde{\mathbf{s}}^{(j)}$.

Reconstruction from incomplete measurements

Theorem (Plonka & Riebe (2024))

Let the incomplete measurement vectors $\mathcal{P}\mathbf{y}^{(j)}$ *for* $j = 0, \ldots, N_c - 1$, *be given, where the index set* $\Lambda_{\mathcal{P}}$ *of acquired measurements contains the calibration region, i.e.,* $\Lambda_{L+M} \subset \Lambda_{\mathcal{P}} \subset \Lambda_N$. *Let the model assumptions for* m *and* $s^{(j)}$ *be satisfied and suppose that:*

1. *The matrix* $\mathbf{A}_{M,L}$ *has a nullspace of dimension* 1.
2. *The matrix* $\sum\limits_{j=0}^{N_c-1} (\mathbf{B}^{(j)})^* \, \mathbf{B}^{(j)}$ *with* $\mathbf{B}^{(j)} := \mathcal{P}\mathcal{F} \, \mathrm{diag}(\tilde{\mathbf{s}}^{(j)})$ *is invertible.*

Then the vector $\boldsymbol{m}$ *determining* m *and the vectors* $\mathbf{c}^{(j)}$ *determining* $s^{(j)}$, $j = 0, \ldots, N_c - 1$, *are uniquely reconstructed by Algorithm 2 up to a constant.*

Theorem (Plonka & Riebe (2024))

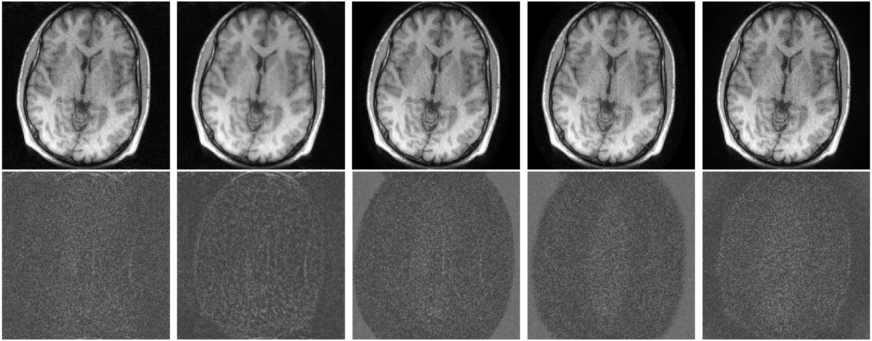
The matrix $\mathbf{A}_{M,L}$ *has a nullspace of dimension* 1 *almost surely.*

Reconstruction results for the first MRI data set

method	measure	$R = 2, (1, \frac{1}{2})$	$R = 3, (1, \frac{1}{3})$	$R = 4, (1, \frac{1}{4})$	$R = 4, (\frac{1}{2}, \frac{1}{2})$	$R = 6, (\frac{1}{2}, \frac{1}{3})$
GRAPPA	PSNR	41.4367	36.2093	30.3212	34.8741	28.9493
	SSIM	0.9638	0.9039	0.7561	0.8821	0.7236
SPIRiT	PSNR	26.7296	28.3496	26.7414	30.8745	29.6801
	SSIM	0.9441	0.8794	0.7192	0.8871	0.7307
ESPIRiT	PSNR	37.2218	35.1977	32.0629	35.4497	32.6245
	SSIM	0.8138	0.7808	0.7022	0.7848	0.7164
L1-ESPIRiT	PSNR	37.3810	35.9555	**34.1941**	35.9088	33.5887
	SSIM	0.8279	0.7689	0.7612	0.7714	0.7589
JSENSE	PSNR	33.5558	34.0555	30.8670		
	SSIM	0.8723	0.8665	0.7818		
MOCCA	PSNR	(12)38.7136	(40)35.1875	(75)32.0755	(50)35.6563	(90)32.0203
(L=5)	SSIM	0.9119	0.8795	0.8111	0.8896	0.7995
MOCCA-S	PSNR	(12)39.4055	(40)36.7334	(75)33.2165	(50)**37.2826**	(90)**33.6516**
(L=5)	SSIM	0.9241	0.9301	**0.8845**	**0.9354**	**0.8897**
MOCCA	PSNR	41.3746.	35.1676	29.1279	35.0675	27.8404
direct (L=5)	SSIM	0.9643	0.8859	0.6491	0.8815	0.6078
MOCCA-S	PSNR	**42.1886**	**37.4517**	32.2398	36.7925	29.3610
direct (L=5)	SSIM	**0.9717**	**0.9369**	0.7876	0.9294	0.7119

Comparison of results of MOCCA algorithm with other state of the art algorithms in parallel MRI for a first data set.

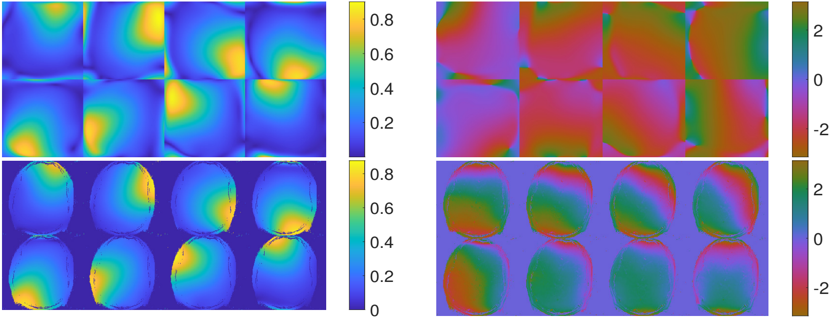
Numerical recovery results for incomplete data



MOCCA MOCCA-S L1-ESPIRiT ESPIRiT GRAPPA

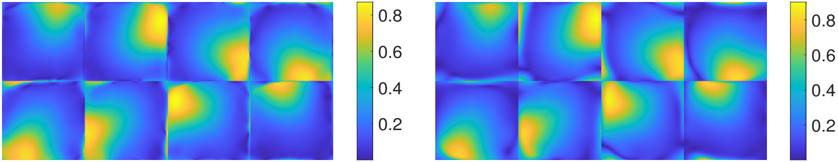
Reconstruction results for the first data set obtained from a third of the k-space data of 8 coils (every third column acquired). All error images use the same scale with relative error in $[0, 0.12]$, where 0 corresponds to black and 0.12 to white.

Reconstruction of coil sensitivities for the first data set



Magnitude (left) and phase (in $[-\pi, \pi]$) (right) of the 8 coil sensitivities obtained for MRI reconstruction for $L = 5$ for the first data set before and after multiplication with $\mathrm{sign}(\boldsymbol{m})$.

Comparison of reconstructed of coil sensitivities



Magnitude of the 8 coil sensitivities obtained for MRI reconstruction (before multiplication with $\mathrm{sign}(\boldsymbol{m})$) for ESPIRiT (left) and for MOCCA with $L = 5$ (right).

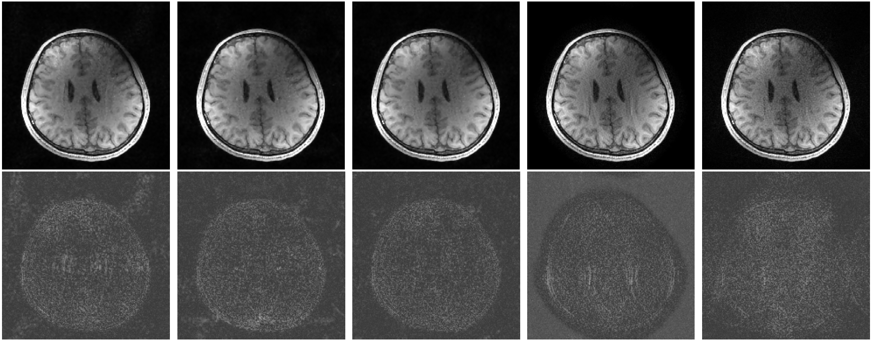
M. Uecker, P. Lai, Mark J. Murphy, P. Virtue, M. Elad, J.M. Pauly, S.S. Vasanawala, and M. Lustig,
ESPIRiT – An eigenvalue approach to autocalibrating parallel MRI: Where SENSE meets GRAPPA,
Magn. Reson. Med. 71(3) (2014), 990–1001.

Numerical results for incomplete MRI data

method	measure	$R=2,(1,\frac{1}{2})$	$R=3,(1,\frac{1}{3})$	$R=4,(1,\frac{1}{4})$	$R=4,(\frac{1}{2},\frac{1}{2})$	$R=6,(\frac{1}{2},\frac{1}{3})$
GRAPPA	PSNR	47.2028	42.9704	37.6036	41.5961	38.5314
	SSIM	0.9796	0.9573	0.9167	0.9498	0.9244
ESPIRiT	PSNR	40.3668	39.7082	37.1418	39.6082	38.4716
	SSIM	0.7490	0.7462	0.7243	0.7461	0.7354
L1-ESPIRiT	PSNR	40.2750	39.7584	38.1926	39.6515	38.8431
	SSIM	0.7465	0.7520	0.7455	0.7521	0.7503
MOCCA	PSNR	(10)42.6611	(50)40.3835	(70)34.8179	(50)40.6865	(90)35.9914
(L=5)	SSIM	0.9258	0.9474	0.8769	0.9472	0.8752
MOCCA-S	PSNR	(10)43.2025	(50)41.7574	(70)35.8326	(50)42.2273	(90)37.5114
(L=5)	SSIM	0.9314	0.9675	0.9192	0.9679	0.9239
MOCCA	PSNR	(10)43.5172	(50)41.9112	(70)37.3703	(50)41.7795	(90)37.4211
(L=7)	SSIM.	0.9251	0.9461	0.9143	0.9534	0.9011
MOCCA-S	PSNR	(10)44.0399	(50)43.5151	(70)38.9200	(50)43.5679	(90)39.3194
(L=7)	SSIM	0.9299	0.9630	0.9537	0.9729	0.9463
MOCCA	PSNR	(10)43.2813	(50)41.7241	(70)38.0292	(50)41.2087	(90)37.9957
(L=9)	SSIM	0.9199	0.9437	0.9214	0.9489	0.9127
MOCCA-S	PSNR	(10)43.7645	(50)43.1347	(70)**39.7013**	(50)42.6955	(90)**40.0862**
(L=9)	SSIM	0.9245	0.9597	**0.9579**	0.9675	**0.9571**
MOCCA	PSNR	47.4233	42.2531	35.8053	42.1402	34.4736
direct (L=7)	SSIM.	0.9844	0.9615	0.8838	0.9616	0.8293
MOCCA-S	PSNR	**47.9400**	**43.8822**	38.0142	**43.9457**	36.4445
direct (L=7)	SSIM	**0.9866**	**0.9764**	0.9374	**0.9792**	0.8954

Comparison of results of MOCCA algorithm with other state of the art algorithms in parallel MRI for a second data set.

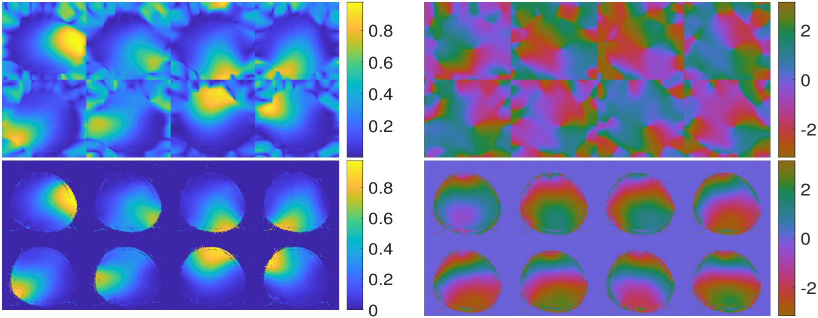
Numerical recovery results for incomplete data



MOCCA-S ($L = 7$)	MOCCA-S ($L = 9$)	MOCCA-S ($L = 11$)	ESPIRiT	GRAPPA

Reconstruction results obtained from a fourth of the k-space data of the second data set with 8 coils (every fourth column acquired).

Reconstruction of coil sensitivities for the second data set



Magnitude (left) and phase (in $[-\pi, \pi]$) (right) of the 8 coil sensitivities obtained for MRI reconstruction for $L = 9$ for the first data set before and after multiplication with $\mathrm{sign}(\boldsymbol{m})$.

Summary

- We have proposed a new model based reconstruction method (MOCCA) for the blind deconvolution problem occurring in parallel MRI.
- The sensitivity functions are modeled by bivariate trigonometric polynomials of small degree and can be recovered in a first step by solving an eigenvalue problem.
- The magnetization image can be recovered from incomplete measurement data by solving a least squares problem.
- The algorithm provides exact recovery up to a global constant if the data satisfy the model.
- The algorithm is numerically stable and highly efficient, it requires $\mathcal{O}(N_c N^2 \log N)$ operations.
- The approach can be simply generalized to other models for the sensitivities.

Publications

Gerlind Plonka, Yannick Riebe,
MOCCA: A fast algorithm for parallel MRI reconstruction using model based coil calibration,
preprint, 2024.

Benjamin Kocurov, Gerlind Plonka,
The mathematical background of MRI subspace methods: ESPIRiT versus MOCCA,
in preparation.