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Solving moment and polynomial optimization problems on Sobolev spaces

SIGMA 2024 at CIRM Luminy

arXiv:2401.07734

Outline

1. POP and moments in finite dimension

2. Towards infinity with Sobolev and Fourier

3. Approximation results

4. Examples

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1. **POP and moments in finite dimension**

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Polynomial optimization problem **(POP)**

$$p^* = \min_{x \in X} p(x)$$

where

$$p = \sum_{a \in \mathbb{N}^n_d} p_a x^a \in \mathbb{R}[x]_d$$

is a given multivariate polynomial of degree d expressed e.g. in the monomial basis

$$x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$$

indexed by $a \in \mathbb{N}^n_d := \{a \in \mathbb{N}^n : |a| := a_1 + a_2 + \cdots + a_n \leq d\}$, and

$$X := \{x \in \mathbb{R}^n : g_i(x) \geq 0, i = 1, 2, \ldots, m\}$$

is a compact basic semialgebraic set described by given multivariate polynomials $g_i \in \mathbb{R}[x]$

Solving POP

$$p^* = \min_{x \in X} \sum_{a \in \mathbb{N}^n_d} p_a x^a$$

is equivalent to solving the linear **moment problem**

$$p^* = \min_{y \in M_d(X)} \sum_{a \in \mathbb{N}^n_d} p_a y_a$$

in the finite-dimensional convex set

$$M_d(X) := \{(y_a)_{a \in \mathbb{N}^n_d} : y_a = \int_X x^a d\mu(x) \text{ for some } \mu \in P(X)\}$$

where $P(X)$ denotes the set of probability measures on X

The truncated moment problem consists of determining whether a given vector $y \in \mathbb{R}^{\mathbb{N}^n_d}$ belongs to

$$M_d(X) := \{y : y_a = \int_X x^a d\mu(x) \text{ for some } \mu \in P(X)\}$$

Example: $n = 1, d = 4, X = [-1, 1]$

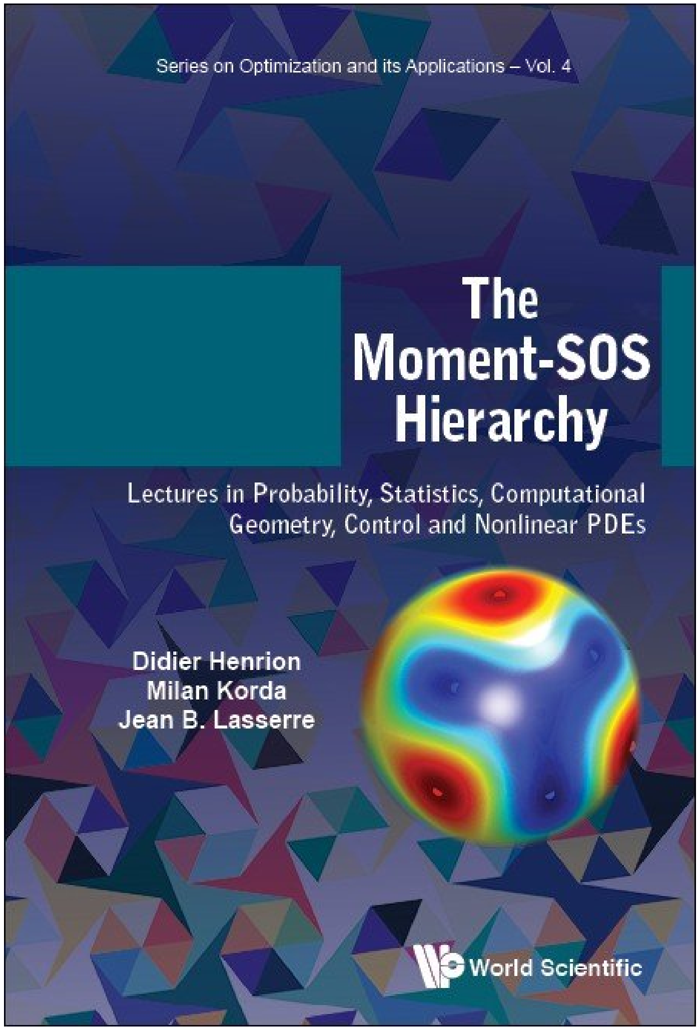
$y = (1, 1, 1, 1, 1) \in M_4(X)$ is represented by $\mu(dx) = \delta_1$:
$y_a = 1^a$, $a = 0, 1, \ldots, 4$

$y = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}) \in M_4(X)$ is represented by $\mu(dx) = \lambda_{[0,1]}(dx)$:
$y_a = \int_0^1 x^a dx = \frac{1}{a+1}$, $a = 0, 1, \ldots, 4$

$y = (1, 1, -1, 1, 1) \notin M_4(X)$ because $\int x^2 d\mu(x) \geq 0$

The truncated moment problem can be solved numerically with semidefinite representable cones constructed with polynomial sums of squares (SOS) of increasing degrees

POP can be solved approximately with the **moment-SOS hierarchy**



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What can be said in **infinite dimensions** ?

How can we make sense of a POP

$$p^* = \min_{x \in X} p(x)$$

if X is a compact set of a Hilbert space $\mathscr{H}$?

Can we solve the moment problem and hence the POP in $\mathscr{H}$ with an infinite-dimensional moment-SOS hierarchy ?

Motivation: optimization and control of polynomial differential equations (ordinary, stochastic, partial) - not covered today

Let $\mathscr{H}$ be a separable Hilbert space with a scalar product $\langle .,. \rangle_{\mathscr{H}}$ and let $(e_k)_{k=1,2,\dots}$ be a complete orthonormal system in $\mathscr{H}$ with $e_1 = 1$

Given $n \in \mathbb{N}$, consider the projection mapping

$$\begin{array}{rccl} \pi_n : & \mathscr{H} & \to & \mathscr{H} \\ & x & \mapsto & \sum_{k=1}^{n} \langle x, e_k \rangle e_k \end{array}$$

Note that

$$|\pi_n(x)|^2 = \sum_{k=1}^{n} \langle x, e_k \rangle^2$$

and

$$|x|^2 = \lim_{n \to \infty} |\pi_n(x)|^2 = \sum_{k=1}^{\infty} \langle x, e_k \rangle^2$$

i.e. $\pi_n(x)$ converges strongly in $\mathscr{H}$ to x when $n \to \infty$

Let $c_0(\mathbb{Z}^n)$ denote the set of integer sequences with finitely many non-zero elements

The **monomial** of degree $a \in c_0(\mathbb{Z}^n)$ is the finite product

$$x^a := \prod_{k=1,2,...} \langle x, e_k \rangle^{a_k}$$

and a **polynomial** in H is a linear combination of monomials

$$p(x) = \sum_{a \in \mathrm{spt}\, p} p_a x^a$$

with scalar coefficients p_a indexed in the support $\mathrm{spt}\, p \subset c_0(\mathbb{Z}^n)$

The **algebraic degree** of $p(.)$ is $d := \max_{a \in \mathrm{spt}\, p} \sum_k a_k$ and the **harmonic degree** of $p(.)$ is $n := \max_{a \in \mathrm{spt}\, p} \{k \in \mathbb{N} : a_k \neq 0\}$

Our Hilbert space $\mathscr{H}$ is the **Sobolev space** of complex functions on the n-dimensional unit torus T^m whose derivatives up to order m are square integrable:

$$H^m(T^m) := \{x \in L^2 : T^n \to \mathbb{C} : \|x\|^2_{H^m(T^n)} < \infty\}$$

where

$$\|x\|^2_{H^m(T^n)} := \sum_{a \in \mathbb{N}^n_m} \int_{T^n} \left\| \frac{\partial^{|a|} x(\theta)}{\partial \theta_1^{a_1} \ldots \partial \theta_n^{a_n}} \right\|_2^2 d\theta$$

Let μ be a Radon measure (i.e. locally finite and tight) on B

The **moment** of μ of index $a \in c_0(\mathbb{Z}^n)$ is

$$y_a := \int_B x^a d\mu(x), \quad x^a := \prod_{k=1,2,\ldots} \langle x, e_k \rangle_{L^2}^{a_k}$$

for the scalar product

$$\langle x, e_k \rangle_{L^2} := \int_{T^n} x(\theta) e_a(\theta) d\theta, \quad e_a(\theta) := e^{-2\pi \mathbf{i} \langle a, \theta \rangle_{\mathbb{R}^n}}$$

Given a set $A \subset c_0(\mathbb{Z}^n)^N$, let us define the Sobolev moment cone

$$C(A) := \{(y_a)_{a \in A} : y_a = \int_B x^a d\mu(x) \text{ for some } \mu\}$$

The **Sobolev truncated moment problem** consists of asking whether a given vector $y \in \mathbb{C}^N$ belongs to $C(A)$

Define the Fourier transform

$$F : L^2(T^n) \to \ell_2(\mathbb{Z}^n),\ \ x \mapsto c$$

with Fourier coefficients

$$c := (c_a)_{a\in\mathbb{Z}^n},\ \ c_a := \langle x, e_a\rangle_{L^2}$$

Its adjoint is the inverse Fourier transform

$$F^* : \ell_2(\mathbb{Z}^n) \to L^2(T^n),\ \ c \mapsto x$$

with

$$x = \langle c, e\rangle_{\ell^2} := \sum_{a\in\mathbb{Z}^n} c_a e_{-a},\ \ e := (e_a)_{a\in\mathbb{Z}^n}.$$

Parseval's identity:

$$\|x\|^2_{H^m(T^n)} = \|c\|^2_W := \sum_{a\in\mathbb{Z}^n} w_a |c_a|^2,\ \ w_a := (1 + \langle a, a\rangle_{\mathbb{Z}^n})^m$$

The Sobolev unit ball

$$B := \{x \in H^m(T^n) : \|x\|_{H^m(T^n)} \leq 1\}$$

becomes an **ellipsoid** in the Fourier coefficients

$$E := \{Fx : x \in B\} = FB = \{c \in \ell_2(\mathbb{Z}^n) : \sum_{a \in \mathbb{Z}^n} w_a |c_a|^2 \leq 1\}$$

which is included in a Hilbert cube

$$E \subset \{c \in \ell_2(\mathbb{Z}^n) : |c_a|^2 \leq \frac{1}{w_a} = \frac{1}{(1 + \langle a, a \rangle_{\mathbb{Z}^n})^m}, a \in \mathbb{Z}^n\}$$

A closed set X is compact if and only if $\forall \epsilon \in \mathbb{R}\, \exists n \in \mathbb{N}$ such that $\sup_{x \in X} |\sum_{k=n+1}^{\infty} \langle x, e_k \rangle e_k|^2 < \epsilon^2$, i.e. uniformly small tail

We use this to prove that E is **compact** if $m \geq 1$

Let $\nu := F_{\#}\mu$ be the image measure of μ through F, i.e.

$$\nu(A) = \mu(\{x \in B : Fx \in A\})$$

for all Borel sets $A \subset E$

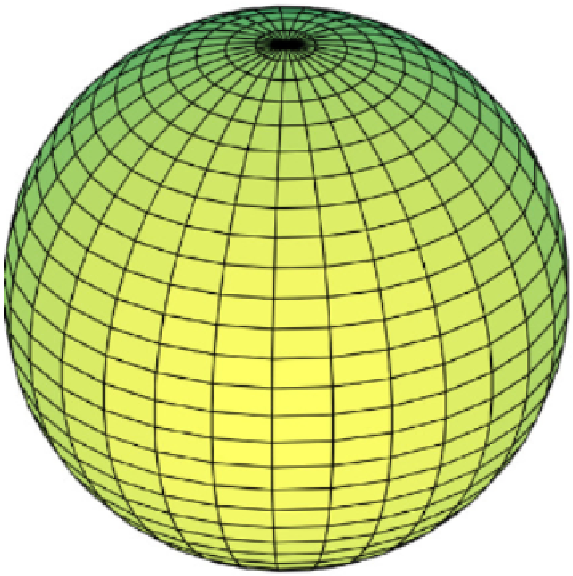
Geometrically, Sobolev moments on the ball B become Fourier moments on the ellipsoid E:

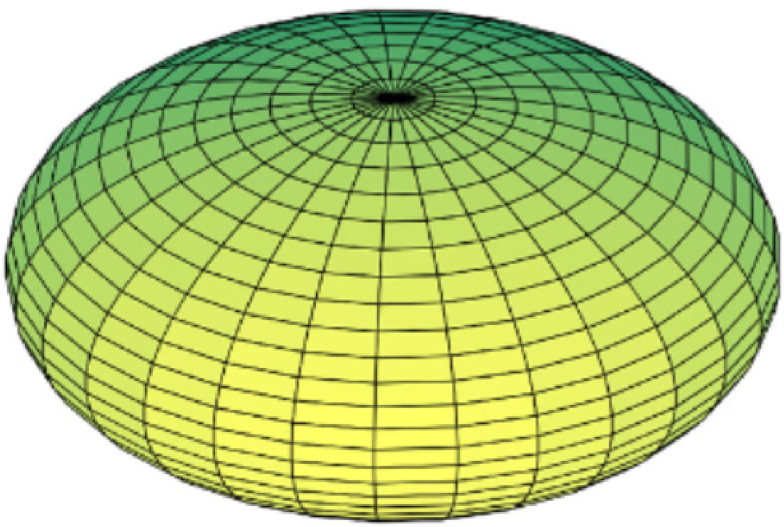
$$y_a = \int_B x^a \, d\mu(x) = \int_E c^a \, d\nu(c)$$

Our Sobolev moment problem is a Fourier moment problem in

$$C(A) := \{(y_a)_{a \in A} : y_a = \int_E c^a d\nu(c) \text{ for some } \nu\}$$

in a compact set E defined on $\mathbb{C}[c]$, the ring of complex polynomials with **countably infinitely many variables** $c = (c_a)_{a \in \mathbb{Z}^n}$





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In complete analogy with the finite-dimensional case, cone $C(A)$ can be approximated by linear sections and projections of a finite-dimensional convex cone on which optimization can be carried out efficiently, namely the semidefinite cone

To construct these approximations and prove their convergence, we use sums of squares (SOS) representations of positive polynomials in the algebra of $\mathbb{C}[(c_a)_{a\in\mathbb{Z}^n}]$

Jacobi's Archimedean Positivstellensatz for commutative unital algebras is Theorem 4 in [T. Jacobi. A representation theorem for certain partially ordered commutative rings. Math. Z. 237:259-273, 2001]

See also Theorem 2.1 in [M. Ghasemi, S. Kuhlmann, M. Marshall. Application of Jacobi's representation theorem to locally multiplicatively convex topological $\mathbb{R}$-algebras. J. Functional Analysis 266:1041-1049, 2014]

See also Theorem 3.9 in [M. Infusino, S. Kuhlmann, T. Kuna, P. Michalski. An intrinsic characterization of moment functionals in the compact case. Int. Math. Res. Not. 2023(3):2281-2303, 2023]

Using Jacobi's Psatz we can approximate the moment cone

$$C(A) = \{(y_a)_{a\in A} : y_a = \int_E c^a \, d\nu(c) \text{ for some } \nu \text{ on } E \} \subset \mathbb{C}^N$$

for a given index set $A \in c_0(\mathbb{Z}^n)$, with a converging **hierarchy of semidefinite representable outer approximations** $C^{\text{out}}_{r,\rho}(A)$ indexed by the algebraic resp. harmonic degree r resp. ρ of the SOS certificates:

$$C^{\text{out}}_{r,\rho}(A) \supset C(A), \quad \overline{C^{\text{out}}_{\infty,\infty}}(A) = C(A)$$

For any $r \geq r_A$ and $\rho \geq \rho_A$, Hausdorff distance bound

$$d_H(C(A), C^{\text{out}}_{r,\rho}(A)) \;\leq\; 9\,(2\rho_A + 1)^n \, \frac{r_A^2}{r^2}$$

Using measures with SOS densities, we can design a converging hierarchy of semidefinite representable inner approx. $C^{\text{inn}}_{r,\rho}(A)$

We are now fully equipped to solve a Sobolev POP

$$p^* := \inf_{x \in B} p(x)$$

with finite harmonic degree $\delta(p)$ and finite algebraic degree $d(p)$

Equivalent to the Fourier POP

$$p^* = \min_{c \in E} p(c)$$

and the linear problem

$$p^* = \min_{\nu \in P(E)} \int_E p(c) d\nu(c) = \min_{\nu \in P(E)} \sum_{a \in A} \int_E c^a \nu(c)$$

on $P(E)$, the probability measures on the Fourier ellipsoid E, in turn equivalent to the linear moment problem

$$p^* = \min_{y \in C(A)} \sum_{a \in A} p_a y_a \text{ s.t. } y_0 = 1$$

Therefore we can design a moment-SOS hierarchy of lower bounds

$$p^{\text{out}}_{r,\rho} := \min_{y \in C^{\text{out}}_{r,\rho}(A)} \sum_{a \in A} p_a y_a$$

as well as a moment-SOS hierarchy of upper bounds

$$p^{\text{inn}}_{r,\rho} := \min_{y \in C^{\text{inn}}_{r,\rho}(A)} \sum_{a \in A} p_a y_a$$

for increasing algebraic resp. harmonic relaxation degrees r, ρ

Theorem: For all $r \geq r' \geq d(p)$ and $\rho = \delta(p)$, it holds

$$p^{\text{out}}_{r',\rho} \leq p^{\text{out}}_{r,\rho} \leq p^{\text{out}}_{\infty,\infty} = p^* = p^{\text{inn}}_{\infty,\rho} \leq p^{\text{inn}}_{r,\rho} \leq p^{\text{inn}}_{r',\rho}$$

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Consider the non-convex **harmonic Sobolev POP**

$$p^* = \min_{x \in B} \langle x, e_0 \rangle^4_{H^0(T)} + (\langle x, e_1 \rangle^2_{H^0(T)} - 1/4)^2$$

on $B \subset H^0(T)$, i.e. $n = 1$, $m = 0$ and harmonic degree $\rho = 1$

Equivalent to the harmonic Fourier POP

$$p^* = \min_{c_{-1}, c_0, c_1} c_0^4 + (c_1^2 - 1/4)^2 \text{ s.t. } c_{-1}^2 + c_0^2 + c_1^2 \leq 1.$$

With the outer moment-SOS hierarchy, at algebraic relaxation degree $r = 2$, we obtain the two global minimizers

$$c^*_{-1} = 0, c^*_0 = 0, c^*_1 = \pm 1/2$$

and the corresponding functions

$$x(\theta) = \pm e^{-2\pi \mathbf{i} \theta}/2$$

achieving the global minimum $p^* = p^{\text{out}}_{2,1} = 0$

Another class is the **algebraic** Sobolev POP

$$p^* = \inf_{x \in B} L(p(x, D^{a_1}x, \ldots, D^{a_l}x))$$

where p is a polynomial of $x \in B$ and its derivatives $D^{a_j}x$, $a_j \in \mathbb{N}^n$ and $L : L^\infty(T^n) \to \mathbb{R}$ is a given bounded linear functional

For example

$$L(p(x)) = \int_{T^n} (p_1(\theta)x(\theta) + p_2(\theta)\|Dx(\theta)\|_2^2) d\sigma(\theta)$$

where σ is a given probability measure on T^n and p_1, p_2 are given real polynomials of θ

The non-linearity hits the function value $x(\theta)$ and its derivatives, and hence this POP generally involves infinitely many harmonics

Consider e.g. the non-convex algebraic Sobolev POP

$$p^* = \inf_{x \in B} \int_T (x(\theta)^2 - 1/2)^2 d\sigma(\theta)$$

where σ is the Dirac measure at 0 on $B \subset H^0(T)$

Since $x(0) = \sum_{a \in \mathbb{Z}} c_a$, the problem is the algebraic Fourier POP

$$p^* = \inf_{c \in E} \frac{1}{4} - \sum_{a_1, a_2 \in \mathbb{Z}} c_{a_1} c_{a_2} + \sum_{a_1, a_2, a_3, a_4 \in \mathbb{Z}} c_{a_1} c_{a_2} c_{a_3} c_{a_4}$$

With the outer moment-SOS hierarchy, at algebraic relaxation degree $r = 2$ and harmonic relaxation degree $\rho = 0$, we obtain the two global minimizers $c_0^* = \pm\sqrt{2}/2$ and the corresponding functions

$$x^*(\theta) = \pm\sqrt{2}/2$$

achieving the global minimum $p^* = p^{\text{out}}_{2,0} = 0$

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