Quadrature rules for splines of high smoothness on uniformly refined triangles

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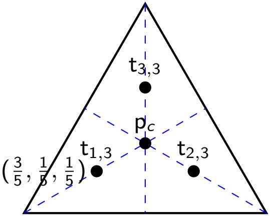
Motivation

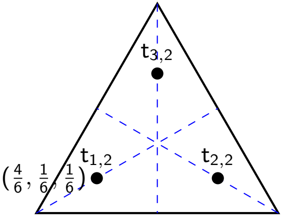
Let $T := \langle \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3 \rangle$ be a non-degenerate triangle. The Hammer-Stroud quadrature rules are defined as (Hammer, Stroud MTAC 1956)

- $Q_{\mathrm{HS},2}[f] := \frac{1}{3}|T|\left(f(\mathrm{t}_{1,2}) + f(\mathrm{t}_{2,2}) + f(\mathrm{t}_{3,2})\right)$
- $Q_{\mathrm{HS},3}[f] := |T|\left(\frac{25}{48}\left(f(\mathrm{t}_{1,3}) + f(\mathrm{t}_{2,3}) + f(\mathrm{t}_{3,3})\right) - \frac{27}{48} f(\mathrm{p}_c)\right)$

satisfy

$$Q_{\mathrm{HS},2}[p] = \int_T p, \quad \forall\, p \in \mathbb{P}_2, \quad \text{and} \quad Q_{\mathrm{HS},3}[p] = \int_T p, \quad \forall\, p \in \mathbb{P}_3$$





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Motivation

- ❓ Is the Hammer-Stroud quadrature rule $Q_{\text{HS},2}$ still exact for the larger space $\mathbb{S}_1$, where $\dim(\mathbb{S}_1) > \dim(\mathbb{P}_2)$?
- ❓ Is the Hammer-Stroud quadrature rule $Q_{\text{HS},3}$ still exact for the larger space $\mathbb{S}_2$, where $\dim(\mathbb{S}_2) > \dim(\mathbb{P}_3)$?

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The answer is positive :

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Gaussian quadrature for C^1 cubic Clough–Tocher macro-triangles

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On numerical quadrature for C^1 quadratic Powell–Sabin 6-split macro-triangles

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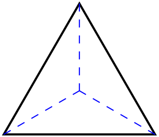
Macro-element splines

Let T_ℓ be a split of a triangle T into ℓ sub-triangles. The spline space of global smoothness r and degree d is defined as:

$$\mathbb{S}_d^r(T_\ell) := \left\{ s \in C^r(T); \quad s_{|\triangle} \in \mathbb{P}_d \quad \forall \triangle \in T_\ell \right\}$$

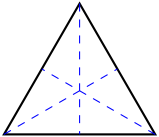
CT-3-split (Clough, Tocher 1965)

$$\#\mathbb{S}_3^1(T_3) = \#\mathbb{P}_3 + 2$$



PS-6-split (Powell, Sabin 1977)

$$\#\mathbb{S}_2^1(T_6) = \#\mathbb{P}_2 + 3$$



Macro-element splines

$\#\mathbb{S}_3^1(T_3) = \#\mathbb{P}_3 + 2$

$\#\mathbb{S}_2^1(T_6) = \#\mathbb{P}_2 + 3$

- $\mathbb{S}_3^1(T_3) = \mathbb{P}_3 \oplus \{D_1, D_2\}$
- $D_i \in C^1$, and $D_{i|\triangle} \in \mathbb{P}_3$
- $D_{i|micro-edge} = 0$, & $\int_T D_i = 0$
  Bartoň, Kosinka 2019

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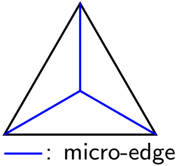

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This technique cannot be generalized to cases involving high degrees or multiple variables.

Simplex spline

Let:

- $V_1, \ldots, V_{d+3} \in \mathbb{R}^2$ be a sequence of knots.
- $V = \{V_1, \ldots, V_{d+3}\} \subset \mathbb{R}^2$.
- $\Theta = \left\langle \widehat{V}_1, \ldots, \widehat{V}_{d+3} \right\rangle$ be a simplex in $\mathbb{R}^{d+2}$.
- $\Pi : \mathbb{R}^{d+2} \longrightarrow \mathbb{R}^2$ be the projection of Θ into $\mathbb{R}^2$ satisfies

$$\Pi\left(\widehat{V}_i\right) = V_i, \qquad i = 1, \ldots, d+3.$$

The unit integral bivariate simplex S_V can be defined geometrically as follows

$$S_V(x) := \frac{\text{vol}_2\left(\Theta \cap \Pi^{-1}(x)\right)}{\text{vol}_{d+2}(\Theta)}$$

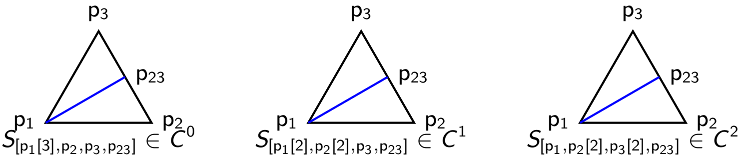
Simplex spline

The simplex S_{V} meets some useful properties:

- S_{V} is a non-negative spline of degree $\leq d$ and support $\langle \mathrm{V} \rangle$.
- For $d = 0$ we have

$$S_{\mathrm{V}} = \begin{cases} \frac{1}{\mathrm{Area}(\langle \mathrm{V} \rangle)}, & \text{if } \mathrm{x} \in \langle \mathrm{V} \rangle^{\circ}, \\ 0, & \text{if } \mathrm{x} \notin \langle \mathrm{V} \rangle^{\circ}, \end{cases}$$

- *Local smoothness:* The simplex S_{V} is $C^{d+1-\mu}$ continuous across a knot line, where μ is the number of knots including multiplicity on that knot line.



$\mathrm{p}[\ell] :=$ p is repeated ℓ times.

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Simplex spline

The simplex S_V meets some useful properties:

- S_V is a non-negative bivariate piecewise polynomial of total degree $\leq d$ and support $\langle V \rangle$.
- For $d = 0$ we have

$$S_V = \begin{cases} \frac{1}{\mathrm{Area}(\langle V \rangle)}, & \text{if } x \in \langle V \rangle^\circ, \\ 0, & \text{if } x \notin \langle V \rangle^\circ, \end{cases}$$

  where $\langle V \rangle^\circ$ marks the open convex hull of the knots sequence V.
- *Local smoothness:* The simplex S_V is $C^{d+1-\mu}$ continuous across a knot line, where μ is the number of knots including multiplicity on that knot line.
- A normalized simplex spline is defined by:

$$N_V := \frac{\mathrm{Area}\,(V)}{\binom{d+2}{2}} S_V$$

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Simplex spline: Knot insertion

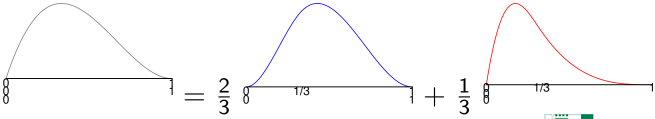
For any $\mathrm{p} \in \mathbb{R}^2$, and any $\beta_1, \ldots, \beta_{d+3} \in \mathbb{R}$ such that $\mathrm{p} = \sum_{i=1}^{d+3} \beta_i \, \mathrm{p}_i$, $\sum_{i=1}^{d+3} \beta_i = 1$, it holds

$$N_{[\mathrm{p}_1,\ldots,\mathrm{p}_{d+3}]} = \sum_{i=1}^{d+3} \beta_i N_{[\mathrm{p}_1,\ldots,\mathrm{p}_{d+3},\mathrm{p}]\setminus \mathrm{p}_i}.$$

Examples:

- 1D:

$$N_{[0[3],1]} = \frac{2}{3} N_{0[2],\frac{1}{3},1]} + \frac{1}{3} N_{0[3],\frac{1}{3}]}$$



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Examples:

- 2D: $\mathrm{p} = \alpha_1 \mathrm{p}_1 + \alpha_2 \mathrm{p}_2 + \alpha_3 \mathrm{p}_3$,

$$N_{[\mathrm{p}_1[4],\mathrm{p}_2[2],\mathrm{p}_3[3]]} = \alpha_1 N_{[\mathrm{p}_1[3],\mathrm{p}_2[2],\mathrm{p}_3[3],\mathrm{p}]} + \alpha_2 N_{[\mathrm{p}_1[4],\mathrm{p}_2,\mathrm{p}_3[3],\mathrm{p}]} + \alpha_3 N_{[\mathrm{p}_1[4],\mathrm{p}_2[2],\mathrm{p}_3[2],\mathrm{p}]}$$

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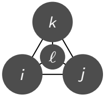
Simplex spline: Notation

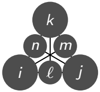
Let $T = \langle \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3 \rangle$ represent a non-degenerate triangle, where $\mathrm{p}_c = \frac{\mathrm{p}_1+\mathrm{p}_2+\mathrm{p}_3}{3}$ denotes the barycenter of T, and $\mathrm{p}_{i,j} = \frac{\mathrm{p}_i+\mathrm{p}_j}{2}$ represents the midpoint of the edge $\langle \mathrm{p}_i, \mathrm{p}_j \rangle$. The following notation is considered:

- CT-3-split:



$$:= N_{[\mathrm{p}_1[i],\mathrm{p}_2[j],\mathrm{p}_3[k],\mathrm{p}_c[\ell]]}$$

- PS-6-split:



$$:= N_{[\mathrm{p}_1[i],\mathrm{p}_2[j],\mathrm{p}_3[k],\mathrm{p}_{1,2}[\ell],\mathrm{p}_{2,3}[m],\mathrm{p}_{3,1}[n]]}$$

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Symmetric quadrature rules

Here we are interested in integrals of the form

$$\int_T f(x)\mathrm{d}x,$$

where f is a given function on the triangle T. Let $n \in \mathbb{N}$, $n > 0$ and a function space $\mathbb{S}$ be given. We denote by

$$\mathcal{Q}_{n,\mathbb{S}}(f) := \text{Area}(T)\sum_{i=1}^{n} \omega_i f(\mathrm{t}_i)$$

an n-node QR that is exact for any function in the space $\mathbb{S}$, i.e.,

$$\mathcal{Q}_{n,\mathbb{S}}(f) = \int_T f(x)\mathrm{d}x \quad \text{for all } f \in \mathbb{S}.$$

The points $\mathrm{t}_i \in T$, are the nodes of $\mathcal{Q}_{n,\mathbb{S}}$ and ω_i are the corresponding weights.

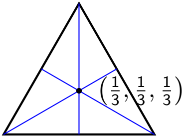
Symmetric quadrature rules

A QR $\mathcal{Q}_{n,\mathbb{S}}$ is said to be symmetric if it maintains its properties under rotation around the barycenter and reflection with respect to the medians of the triangle. Namely, if $\mathrm{t}\,(\alpha_1, \alpha_2, \alpha_3)$ is a node of $\mathcal{Q}_{n,\mathbb{S}}$ with the corresponding weight ω, then for every permutation Π of $(\alpha_1, \alpha_2, \alpha_3)$, the node $\mathrm{t}_\Pi\,(\Pi\,(\alpha_1, \alpha_2, \alpha_3))$ is also a node of $\mathcal{Q}_{n,\mathbb{S}}$ with the same weight ω.

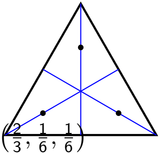
- type-0-orbit $\mathrm{t}\,(1/3, 1/3, 1/3)$
- type-1-orbit $\mathrm{t}\,(1-2\alpha, \alpha, \alpha)$
- type-2-orbit $\mathrm{t}\,(\alpha, \beta, \gamma = 1-\alpha-\beta)$ $(\alpha \neq \beta \neq \gamma)$

Symmetric quadrature rules

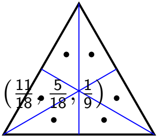
- type-0-orbit t $(1/3, 1/3, 1/3)$



- type-1-orbit t $(1 - 2\alpha, \alpha, \alpha)$



- type-2-orbit t $(\alpha, \beta, \gamma = 1 - \alpha - \beta)$ $(\alpha \neq \beta \neq \gamma)$



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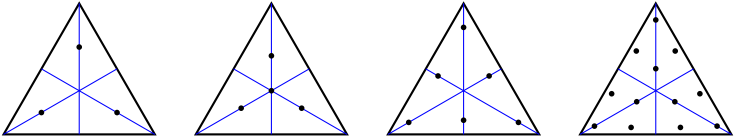
Symmetric quadrature rules

The notation $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{S}}$ represents the QR that uses:

- n_0 type-0 orbits,
- n_1 type-1 orbits,
- n_2 type-2 orbits.

The total number of nodes in this QR is given by

$$n = n_0 + 3n_1 + 6n_2.$$



$\mathcal{Q}_{[0,1,0],\mathbb{P}_2}$ $\mathcal{Q}_{[1,1,0],\mathbb{P}_3}$ $\mathcal{Q}_{[0,2,0],\mathbb{P}_4}$ $\mathcal{Q}_{[0,2,1],\mathbb{P}_5}$

CT-3-split

For any $0 \leq \rho < d$, we have

$$\dim \mathbb{S}_d^\rho(T_3) = \binom{\rho+2}{2} + 3\binom{d-\rho+1}{2} + \sum_{j=1}^{d-\rho} \max\{\rho+1-2j, 0\}.$$

It holds:

$$\#\mathbb{S}_{3r}^{2r-1}(T_3) = \#\mathbb{P}_{3r} + 2.$$

Examples:

$$\mathbb{S}_3^1(T_3) = \#\mathbb{P}_3 + 2, \quad \mathbb{S}_6^3(T_3) = \#\mathbb{P}_6 + 2, \quad \mathbb{S}_9^5(T_3) = \#\mathbb{P}_9 + 2.$$

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CT-3-split: Simplex spline basis for $\mathbb{S}_3^1(T_3)$

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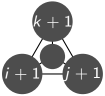
- $\mathbb{P}_3 = \text{span}\Bigg\{$ 

$, \quad i + j + k = 3 \Bigg\}$

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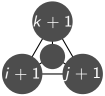
- $\mathbb{P}_3 = \text{span}\Big\{$ 

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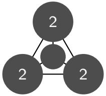
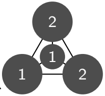
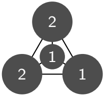
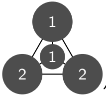
- Knot insertion:



$= \dfrac{1}{3}\Big($ 

$+$ 

$+$ 

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CT-3-split: Simplex spline basis for $\mathbb{S}_3^1(T_3)$

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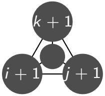
- $\mathbb{P}_3 = \text{span}\Big\{$ 

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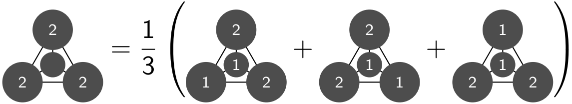
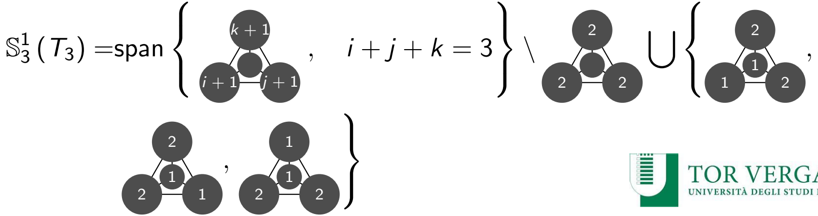
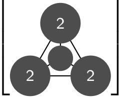


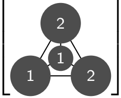
- 

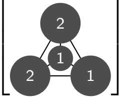
CT-3-split: Simplex spline basis for $\mathbb{S}_3^1(T_3)$

Theorem

Any quadrature rule $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_3}$ is exact on the spline space $\mathbb{S}_3^1(T_3)$.

- $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_3}$ is exact on cubic polynomials
- $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_3}$ [

] $=$ $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_3}$ [

] $=$ $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_3}$ [

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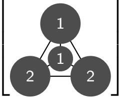
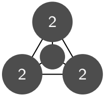
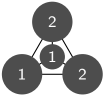
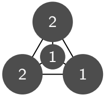
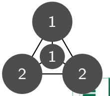
$\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_3}$ [

]
- $\int_T$ 

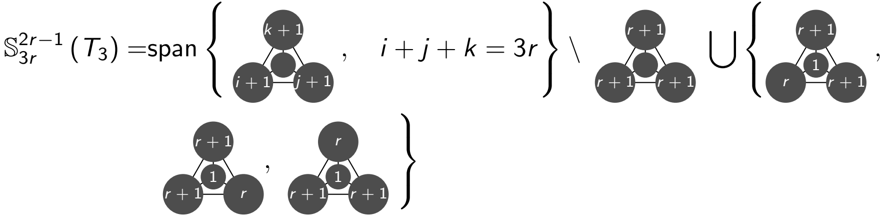
$= \int_T$ 

$= \int_T$ 

$= \int_T$ 

CT-3-split: Simplex spline basis for $\mathbb{S}_{3r}^{2r-1}(T_3)$

$$\#\mathbb{S}_{3r}^{2r-1}(T_3) = \#\mathbb{P}_{3r} + 2$$



Theorem

Any quadrature rule $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_{3r}}$ is exact on the spline space $\mathbb{S}_{3r}^{2r-1}(T_3)$.

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PS-6-split

For any $0 \leq \rho < d$, we have

$$\dim \mathbb{S}_d^{\rho}(T_6) = \binom{\rho+2}{2} + 6\binom{d-\rho+1}{2} + \sum_{j=1}^{d-\rho} \max\{\rho+1-2j, 0\}.$$

It holds:

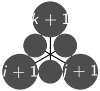
$$\#\mathbb{S}_{2r}^{2r-1}(T_6) = \#\mathbb{P}_{2r} + 3.$$

Examples:

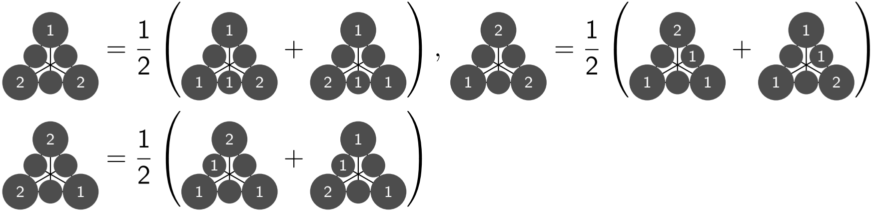
$$\mathbb{S}_2^1(T_6) = \#\mathbb{P}_2 + 3, \quad \mathbb{S}_4^3(T_6) = \#\mathbb{P}_4 + 3, \quad \mathbb{S}_6^5(T_6) = \#\mathbb{P}_6 + 3.$$

PS-6-split: Simplex spline basis for $\mathbb{S}_2^1(T_6)$

$$\#\mathbb{S}_2^1(T_6) = \#\mathbb{P}_2 + 3$$

- $\mathbb{P}_2 = \text{span}\Bigg\{$ 

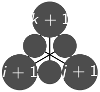
$,\quad i+j+k=2 \Bigg\}$
- Knot insertion:



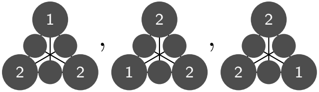
PS-6-split: Simplex spline basis for $\mathbb{S}_2^1(T_6)$

$$\#\mathbb{S}_2^1(T_6) = \#\mathbb{P}_2 + 3$$

$\mathbb{S}_2^1(T_6) = \text{span}\Big\{$

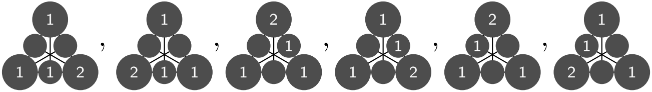


$,\quad i + j + k = 2 \Big\} \setminus \Big\{$



$\Big\}$

$\bigcup \Big\{$

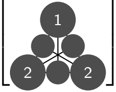


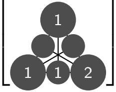
$\Big\}$

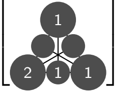
PS-6-split: Simplex spline basis for $\mathbb{S}_2^1(T_6)$

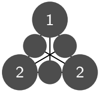
Theorem

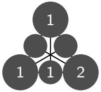
Any quadrature rule $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_2}$ is exact on the spline space $\mathbb{S}_2^1(T_6)$.

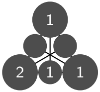
- $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_2}$ is exact on quadratic polynomials
- $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_2}$ [

] $= \mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_2}$ [

] $= \mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_2}$ [

]
- $\int_T$ 

$= \int_T$ 

$= \int_T$ 

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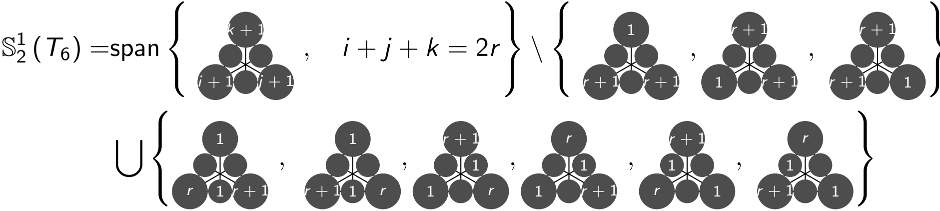
PS-6-split: Simplex spline basis for $\mathbb{S}_{2r}^{2r-1}(T_6)$

$$\#\mathbb{S}_{2r}^{2r-1}(T_6) = \#\mathbb{P}_{2r} + 3$$



Theorem

Any quadrature rule $\mathcal{Q}_{[n_0,n_1,n_2],\mathbb{P}_{2r}}$ is exact on the spline space $\mathbb{S}_{2r}^{2r-1}(T_6)$.

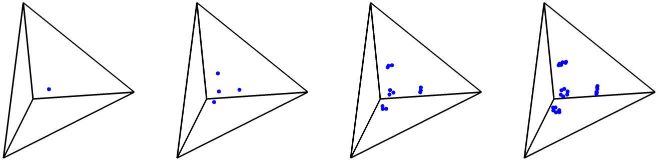
Splines on uniformly refined tetrahedra

Could we apply similar techniques to trivariate splines?

Symmetric quadrature rules in $\mathbb{R}^3$

If $t(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a node of $\mathcal{Q}_{n,\mathbb{S}}$ with the corresponding weight ω, then for every permutation Π of $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, the node $t_\Pi(\Pi(\alpha_1, \alpha_2, \alpha_3, \alpha_4))$ is also a node of $\mathcal{Q}_{n,\mathbb{S}}$ with the same weight ω.

- type-0-orbit $t(1/4, 1/4, 1/4, 1/4)$
- type-1-orbit $t(1-3\alpha, \alpha, \alpha, \alpha)$
- type-2-orbit $t(1-2\alpha-\beta, \beta, \alpha, \alpha)$
- type-3-orbit $t(\alpha, \beta, \gamma, \delta = 1-\alpha-\beta-\gamma)$ $(\alpha \neq \beta \neq \gamma \neq \delta)$



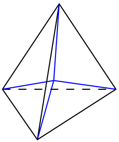
Macro-element splines

Let T_ℓ be a split of a tetrahedra $T = \langle \mathrm{p}_1, \mathrm{p}_2, \mathrm{p}_3, \mathrm{p}_4 \rangle$ into ℓ sub-tetrahedra. The spline space of trivariate global smoothness r and degree d is defined as:

$$\mathbb{S}_d^r(T_\ell) := \left\{ s \in C^r(T); \quad s_{|\triangle} \in \mathbb{P}_d \quad \forall \triangle \in T_\ell \right\}$$

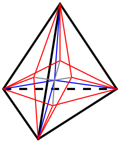
AL-4-split (Alfeld 1984)

$\#\mathbb{S}_4^1(T_4) = \#\mathbb{P}_4 + 3$



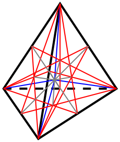
FW-12-split (Farin Worsey 1987)

$\#\mathbb{S}_3^1(T_{12}) = \#\mathbb{P}_3 + 8$



WP-24-split (Worsey-Piper 1988)

$\#\mathbb{S}_2^1(T_{24}) = \#\mathbb{P}_2 + 6$



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AL-4- ($\mathbb{S}_4^1(T_4)$), FW-12- ($\mathbb{S}_3^1(T_{12})$) & WP-24-split ($\mathbb{S}_2^1(T_{24}$

Theorem

- *Any quadrature rule* $\mathcal{Q}_{[n_0,n_1,n_2,n_3],\mathbb{P}_4}$ *is exact on the spline space* $\mathbb{S}_4^1(T_4)$ *(Alfeld).*
- *Any quadrature rule* $\mathcal{Q}_{[n_0,n_1,n_2,n_3],\mathbb{P}_3}$ *is exact on the spline space* $\mathbb{S}_3^1(T_{12})$ *(Farin-Worsey).*
- *Any quadrature rule* $\mathcal{Q}_{[n_0,n_1,n_2,n_3],\mathbb{P}_2}$ *is exact on the spline space* $\mathbb{S}_2^1(T_{24})$ *(Worsey-Piper).*

AL-4- ($\mathbb{S}_4^1(T_4)$), FW-12- ($\mathbb{S}_3^1(T_{12})$) & WP-24-split ($\mathbb{S}_2^1(T_{24}$

Theorem

- *Any quadrature rule* $\mathcal{Q}_{[n_0,n_1,n_2,n_3],\mathbb{P}_4}$ *is exact on the spline space* $\mathbb{S}_4^1(T_4)$ *(Alfeld).*
- *Any quadrature rule* $\mathcal{Q}_{[n_0,n_1,n_2,n_3],\mathbb{P}_3}$ *is exact on the spline space* $\mathbb{S}_3^1(T_{12})$ *(Farin-Worsey).*
- *Any quadrature rule* $\mathcal{Q}_{[n_0,n_1,n_2,n_3],\mathbb{P}_2}$ *is exact on the spline space* $\mathbb{S}_2^1(T_{24})$ *(Worsey-Piper).*

Thank you

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