Analysis and numerics of nonlinear PDE systems in porous media flow models

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Motivation

Standard setup: Flow in a porous medium in domain $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$, with permeability a, flux v, effective pressure u:

$$\begin{aligned} v + a\nabla u &= 0 \quad \text{(Darcy's law)} \\ \nabla \cdot v &= 0 \quad \text{(mass conservation)} \\ &\text{+ boundary conditions} \end{aligned}$$

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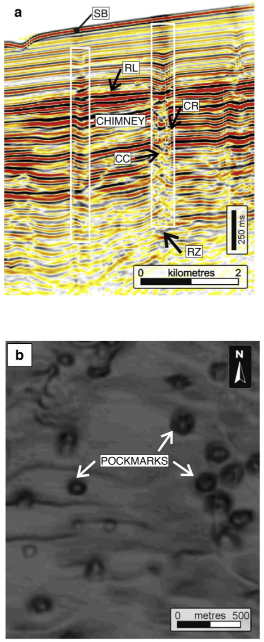
- Porosity can spontaneously form solitary waves or channels (e.g., rising magma),
- Implications for geoengineering: e.g., reservoir safety, CO_2 sequestration, geothermal energy,
- Pockmarks: observed in seabed sediments as precursors to earthquakes. (Christodoulou et al. 2003)



(in Räss et al. 2018)

Full system of PDEs

Poroviscoelastic model in terms of porosity $\phi \in (0,1)$ and effective pressure u

(Conolly, Podladchikov 1998; Vasilyev et al. 2001; …)

On domain $\Omega \subset \mathbb{R}^d$, $d \geq 1$,

$$\begin{aligned}
\partial_t \phi &= -(1-\phi)\left(\frac{b(\phi)}{\sigma(u)}u + Q\partial_t u\right), \\
\partial_t u &= \frac{1}{Q}\left(\nabla \cdot a(\phi)(\nabla u + (1-\phi)g) - \frac{b(\phi)}{\sigma(u)}u\right),
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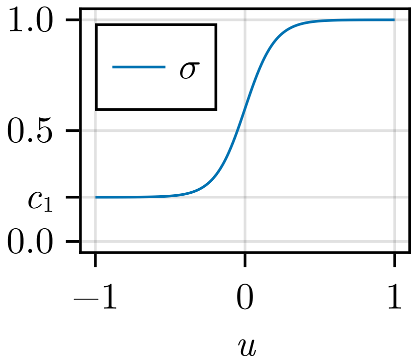
where

- $a(\phi) = a_0\phi^n$ with $n \in [2,4]$, $b(\phi) = \phi^m$ with $m \geq 1$, $g \in \mathbb{R}^d$ and $Q > 0$ constant, σ monotonically increasing and positive (non-constant σ modelling decompaction weakening),



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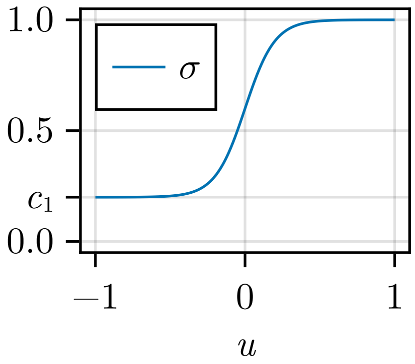
  $g \in \mathbb{R}^d$ and $Q > 0$ constant,
  σ monotonically increasing and positive
  (non-constant σ modelling decompaction weakening),
- Initial conditions $\phi|_{t=0} = \phi_0$, $u|_{t=0} = u_0$,
  Dirichlet or Neumann boundary conditions on u.



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Common simplifications

- Viscous limit case $Q = 0$,

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\begin{aligned}
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\end{aligned}
\quad \longrightarrow \quad
\begin{aligned}
\partial_t \phi &= -(1-\phi)\frac{b(\phi)}{\sigma(u)}u \\
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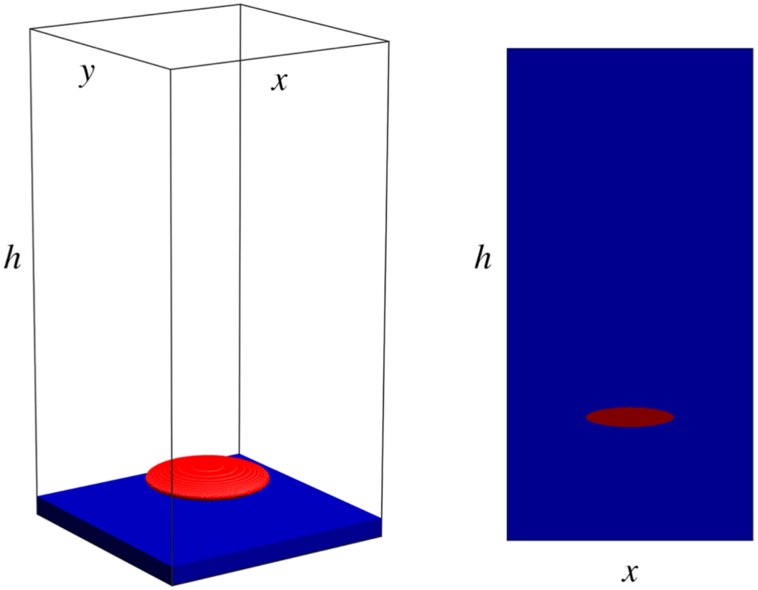
- Low-porosity approximation $1 - \phi \approx 1$,

$$\begin{aligned} \partial_t \phi &= -(1-\phi)\left(\frac{b(\phi)}{\sigma(u)}u + Q\partial_t u\right) \\ \partial_t u &= \frac{1}{Q}\left(\nabla \cdot a(\phi)(\nabla u + (1-\phi)g) - \frac{b(\phi)}{\sigma(u)}u\right) \end{aligned} \quad \longrightarrow \quad \begin{aligned} \partial_t \phi &= -\left(\frac{b(\phi)}{\sigma(u)}u + Q\partial_t u\right) \\ \partial_t u &= \frac{1}{Q}\left(\nabla \cdot a(\phi)(\nabla u + g) - \frac{b(\phi)}{\sigma(u)}u\right) \end{aligned}$$

Formation of channels

Typically of interest: nonsmooth initial porosity ϕ_0



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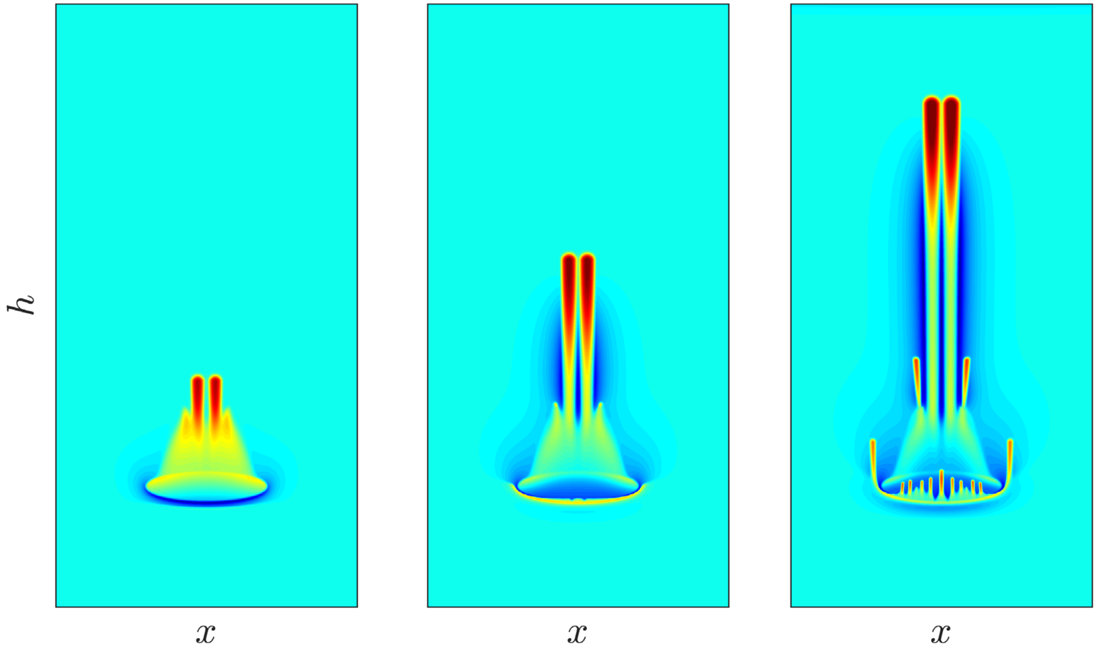
Formation of channels

Results for purely viscous model ($Q = 0$) including shear stresses



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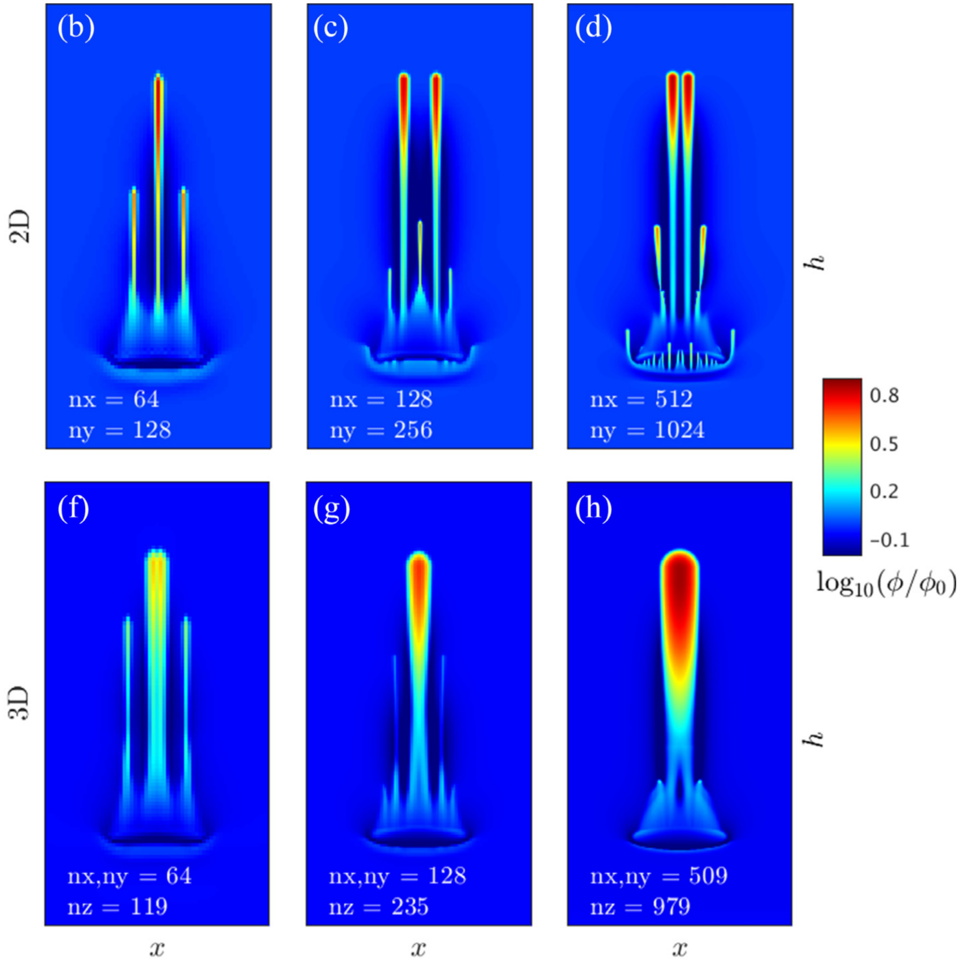
Convergence?



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Basic difficulties with nonsmooth porosities

$$\begin{aligned}\partial_t \phi &= -\boxed{(1-\phi)}\left(\frac{b(\phi)}{\sigma(u)}u + Q\boxed{\partial_t u}\right),\\ \partial_t u &= \frac{1}{Q}\left(\nabla \cdot a(\phi)(\nabla u + (1-\phi)g) - \frac{b(\phi)}{\sigma(u)}u\right).\end{aligned}$$

- First issue: when $1-\phi(t,\cdot) \in L^\infty(\Omega)$ and $\partial_t u(t,\cdot) \in H^{-1}(\Omega)$, how to make sense of $\boxed{(1-\phi)\partial_t u}$?

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$$\partial_t \phi = -(1-\phi)\left(\frac{b(\phi)}{\sigma(u)}u + Q\,\partial_t u\right),$$
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- First issue: when $1-\phi(t,\cdot) \in L^\infty(\Omega)$ and $\partial_t u(t,\cdot) \in H^{-1}(\Omega)$, how to make sense of $(1-\phi)\partial_t u$?
- This problem disappears in the low-porosity approximation $1-\phi \approx 1$,

$$\partial_t \phi = -\left(\frac{b(\phi)}{\sigma(u)}u + Q\partial_t u\right),$$
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- Commonly used, but can lead to unphysical solutions with $\phi > 1$.

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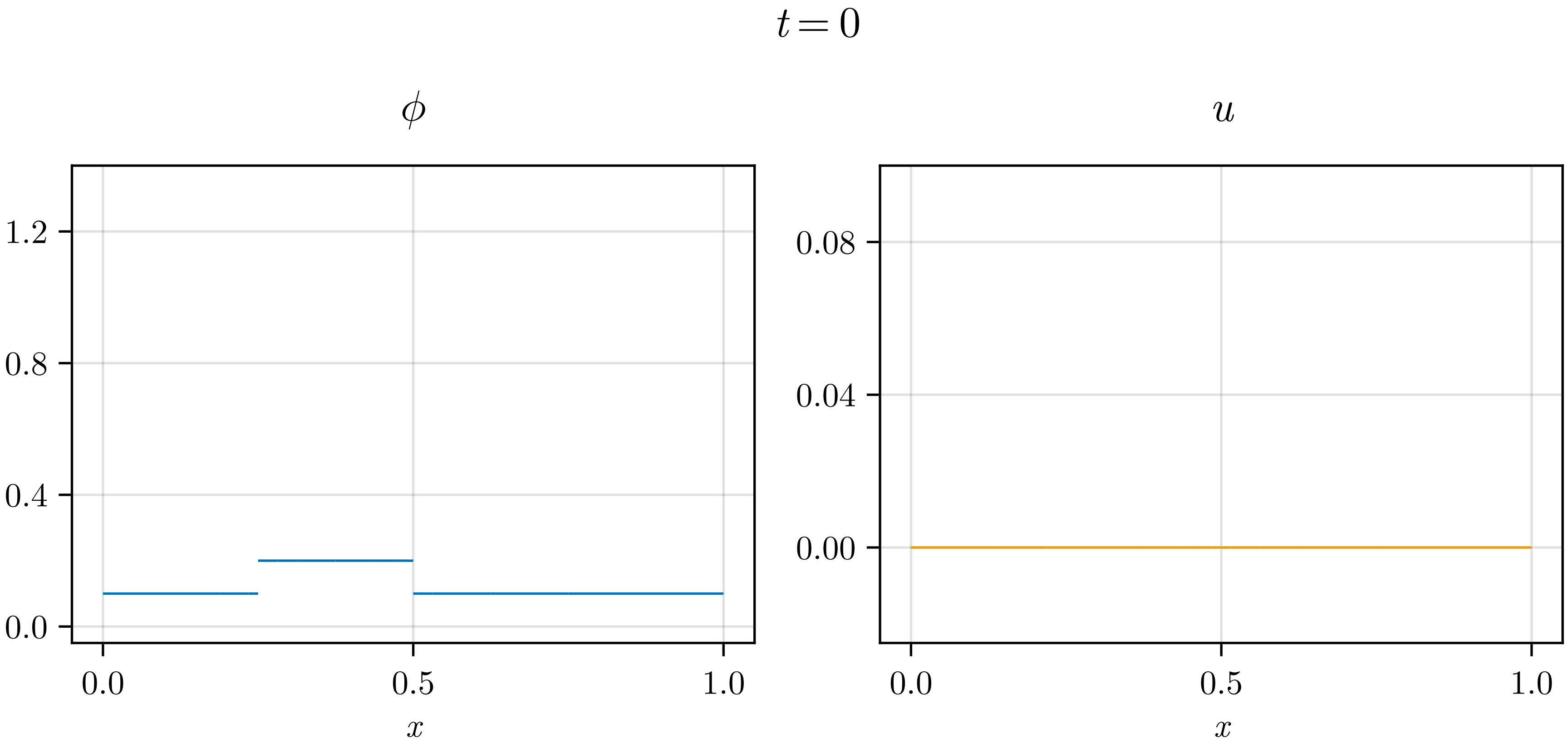

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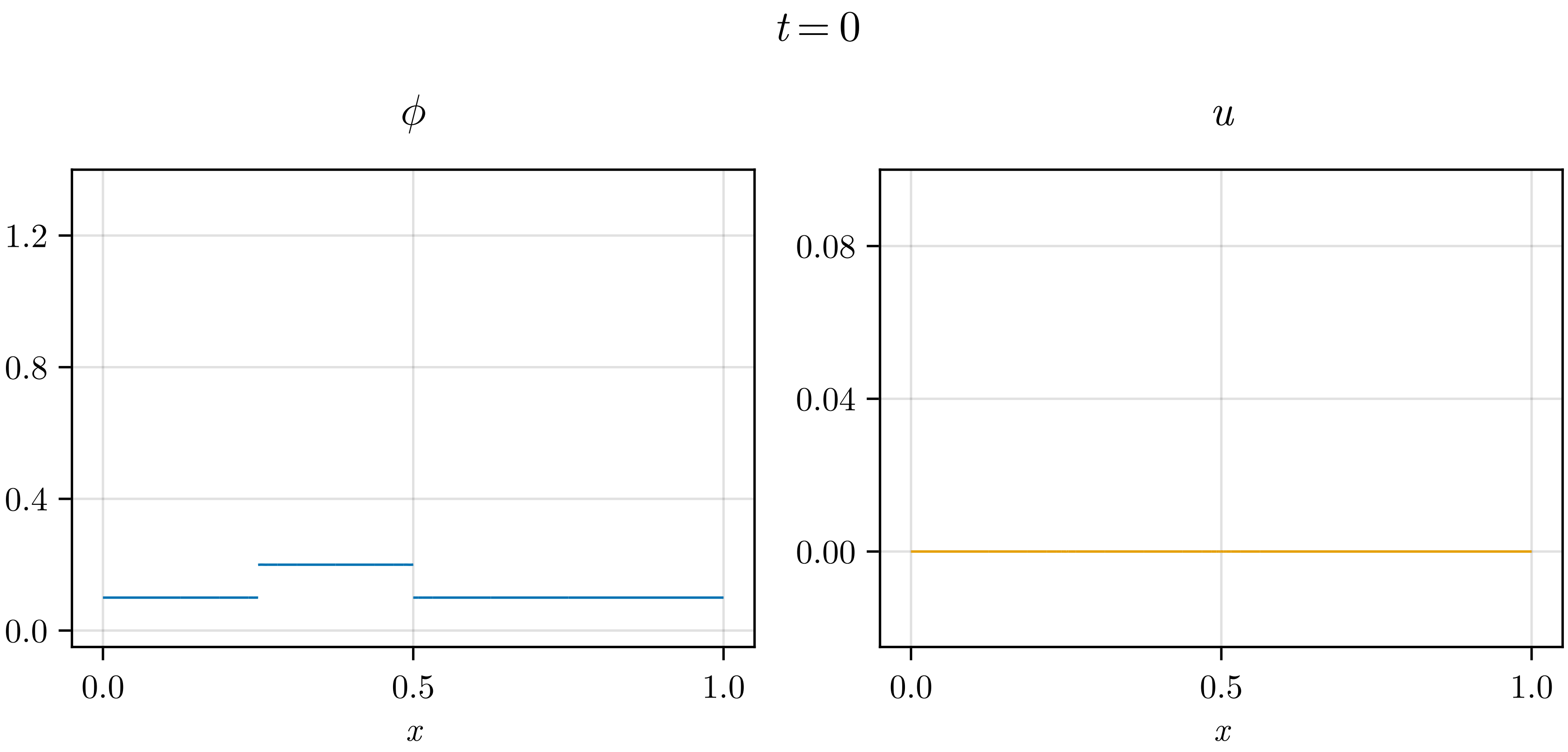

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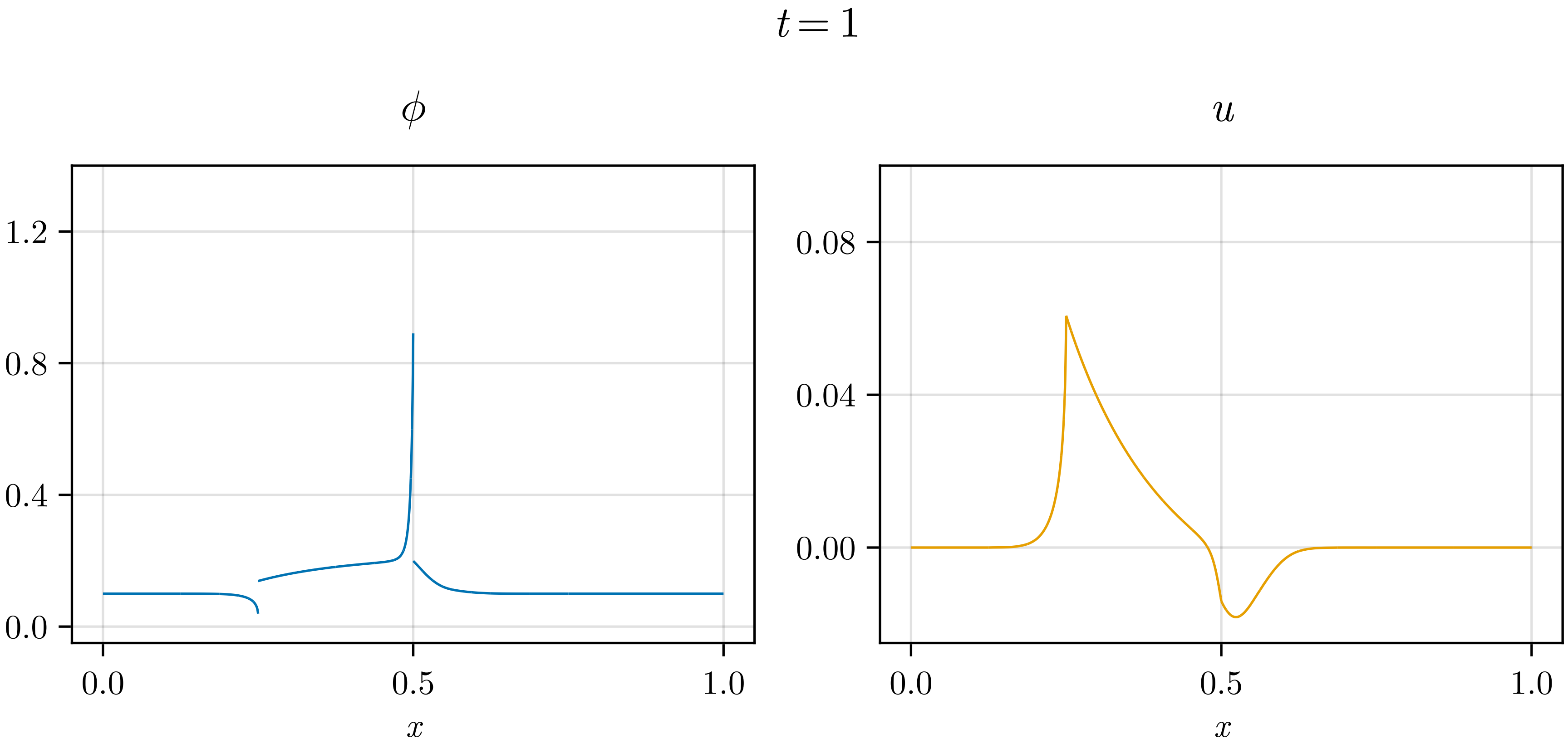

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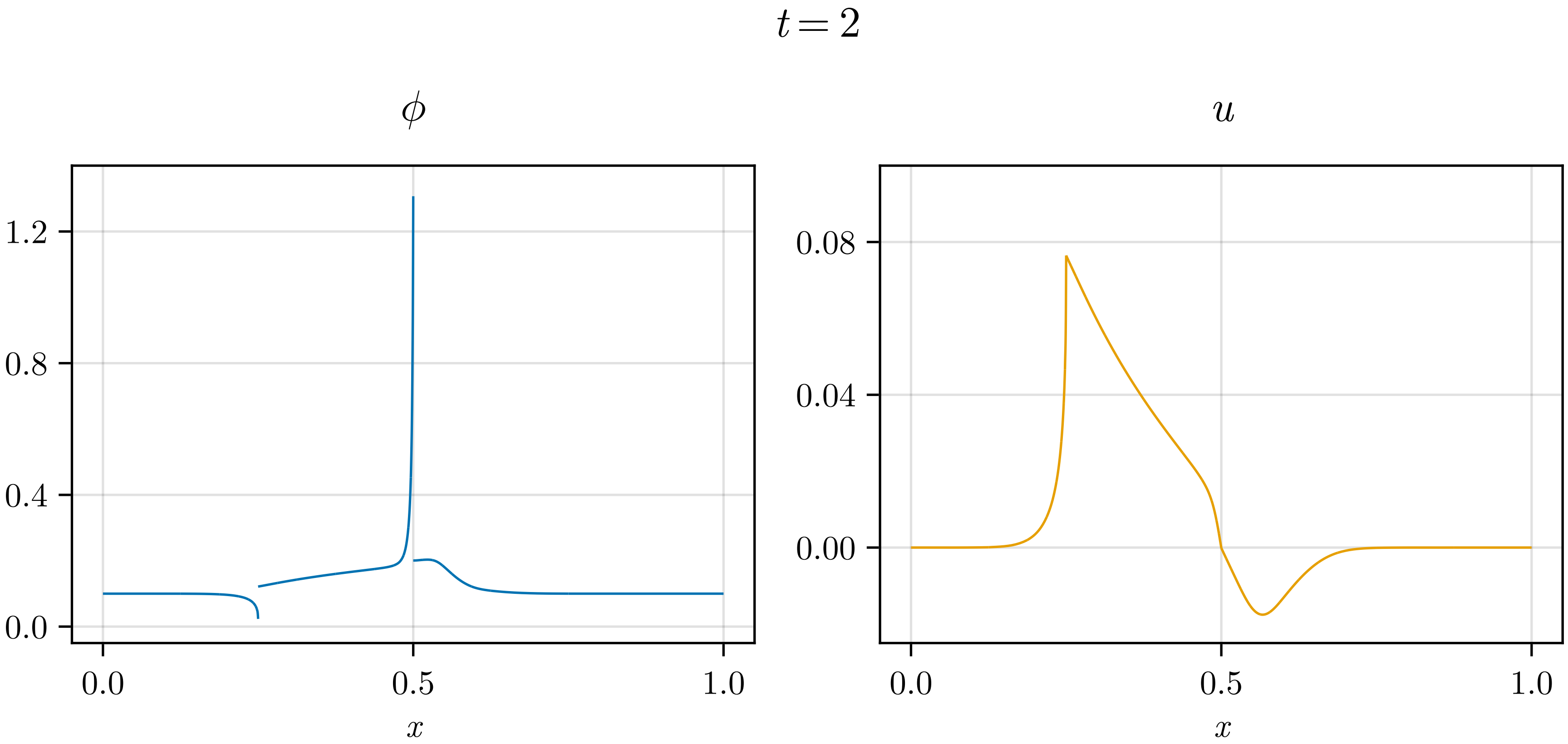

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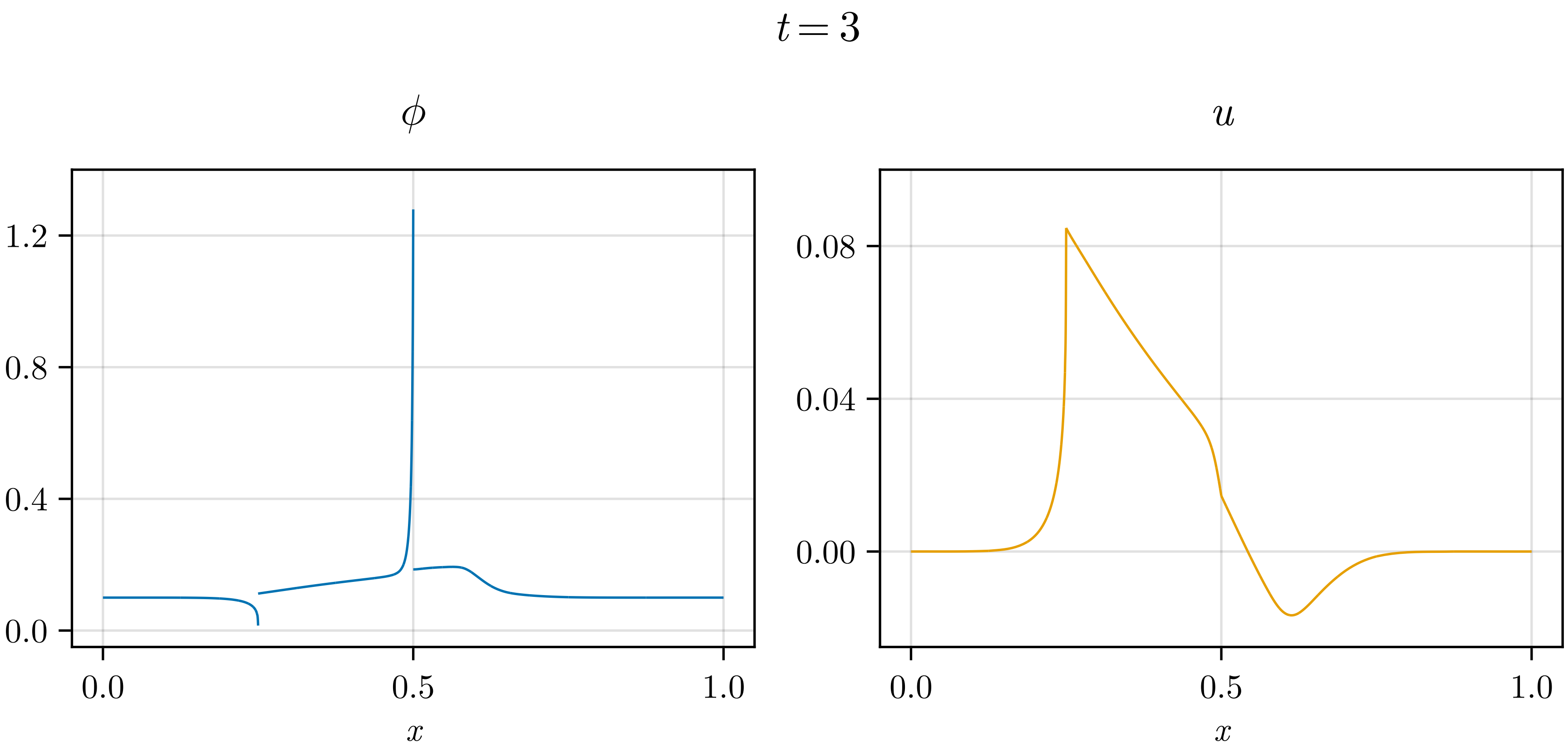

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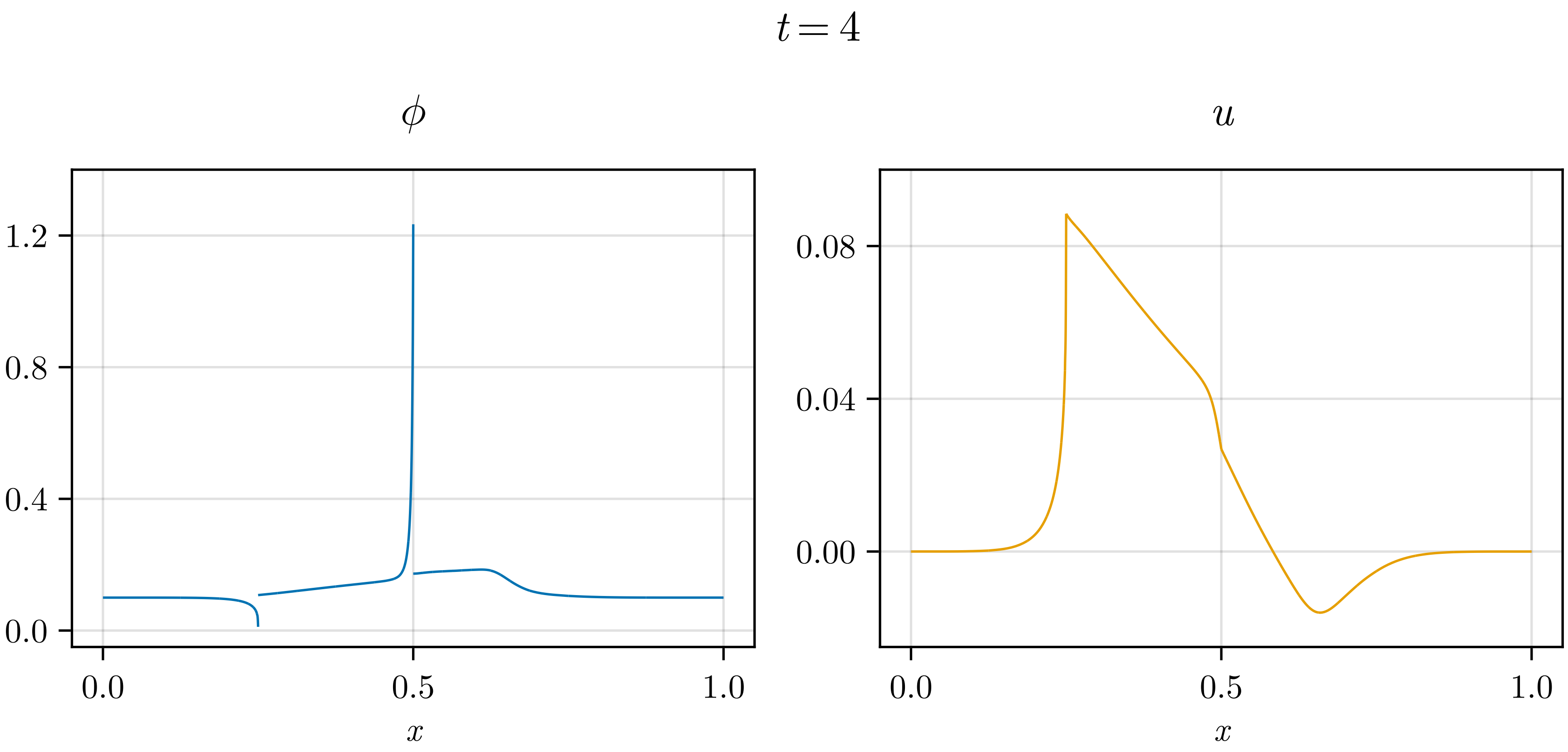

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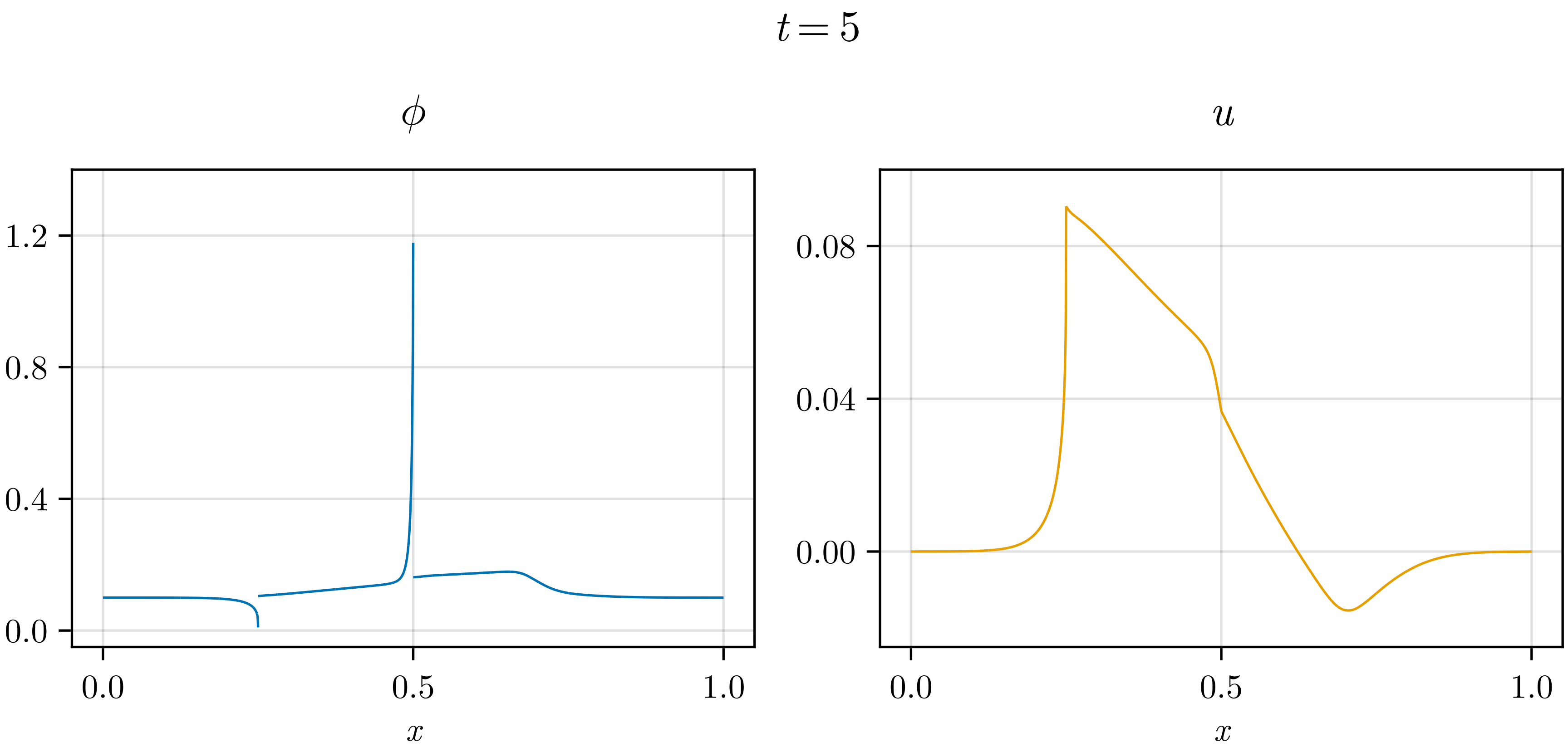

Logarithmic derivative

- Rewrite equation for ϕ with logarithmic derivative on left hand side

  $\rightsquigarrow$ smooth transformation preserves regularity.

- With $\lambda := -\log(1-\phi)$, i.e. $\phi = 1 - e^{-\lambda}$, solve instead

$$\partial_t \lambda = -\left(\frac{b(1-e^{-\lambda})}{\sigma(u)} u + Q \partial_t u \right),$$

$$\partial_t u = \frac{1}{Q} \left(\nabla \cdot a(1-e^{-\lambda})(\nabla u + e^{-\lambda} g) - \frac{b(1-e^{-\lambda})}{\sigma(u)} u \right).$$

  Note that $\phi \in (0,1)$ precisely when $\lambda \in (0,\infty)$.

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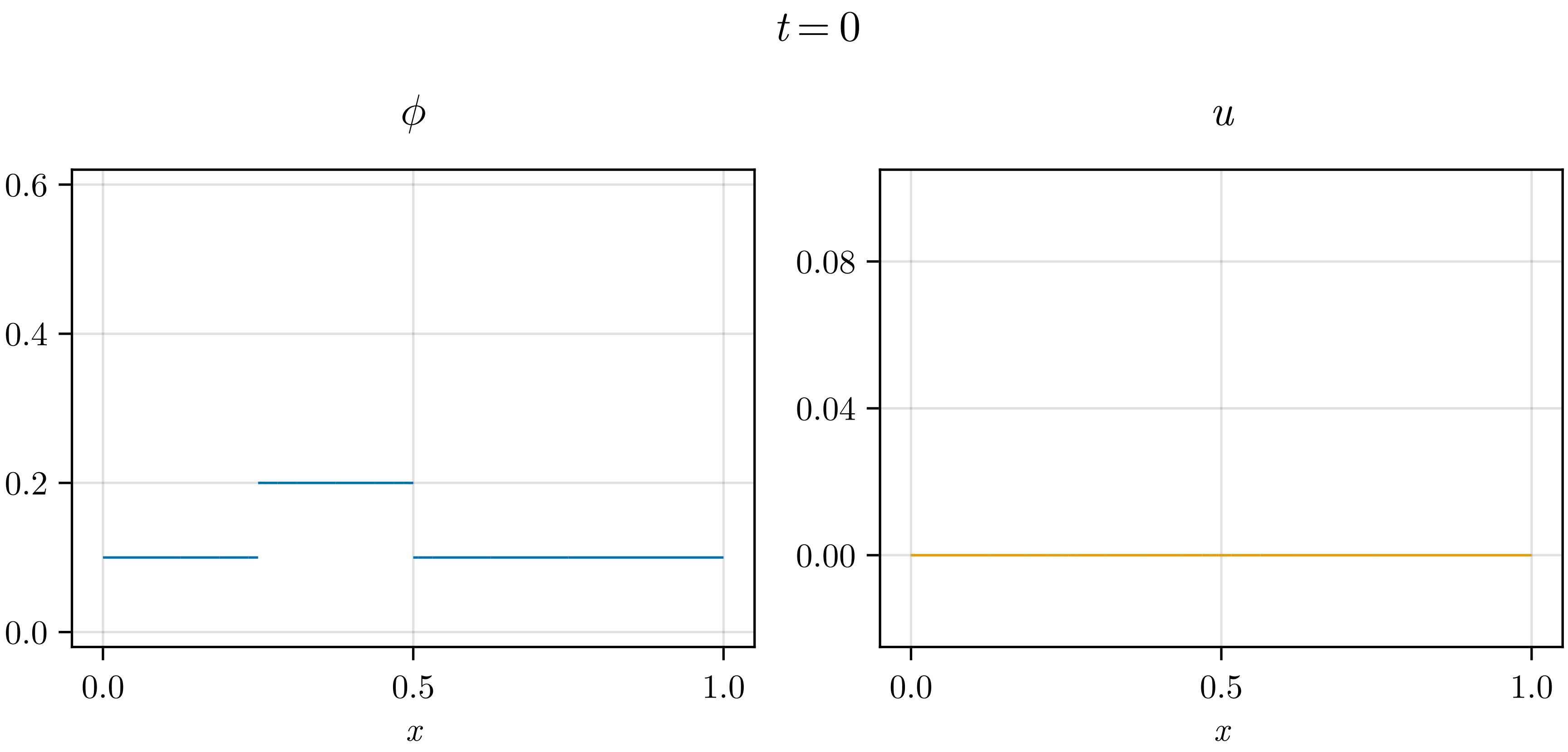

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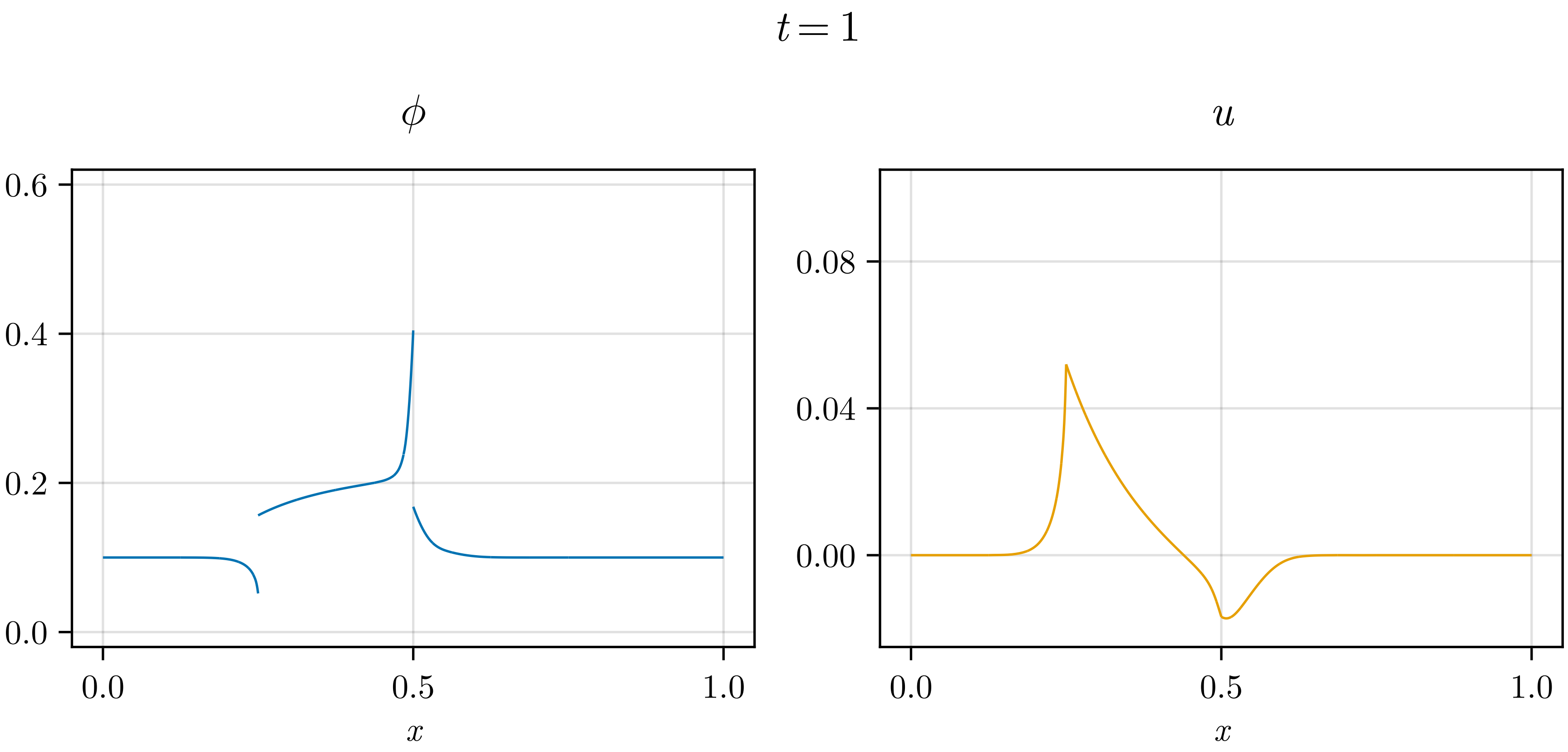

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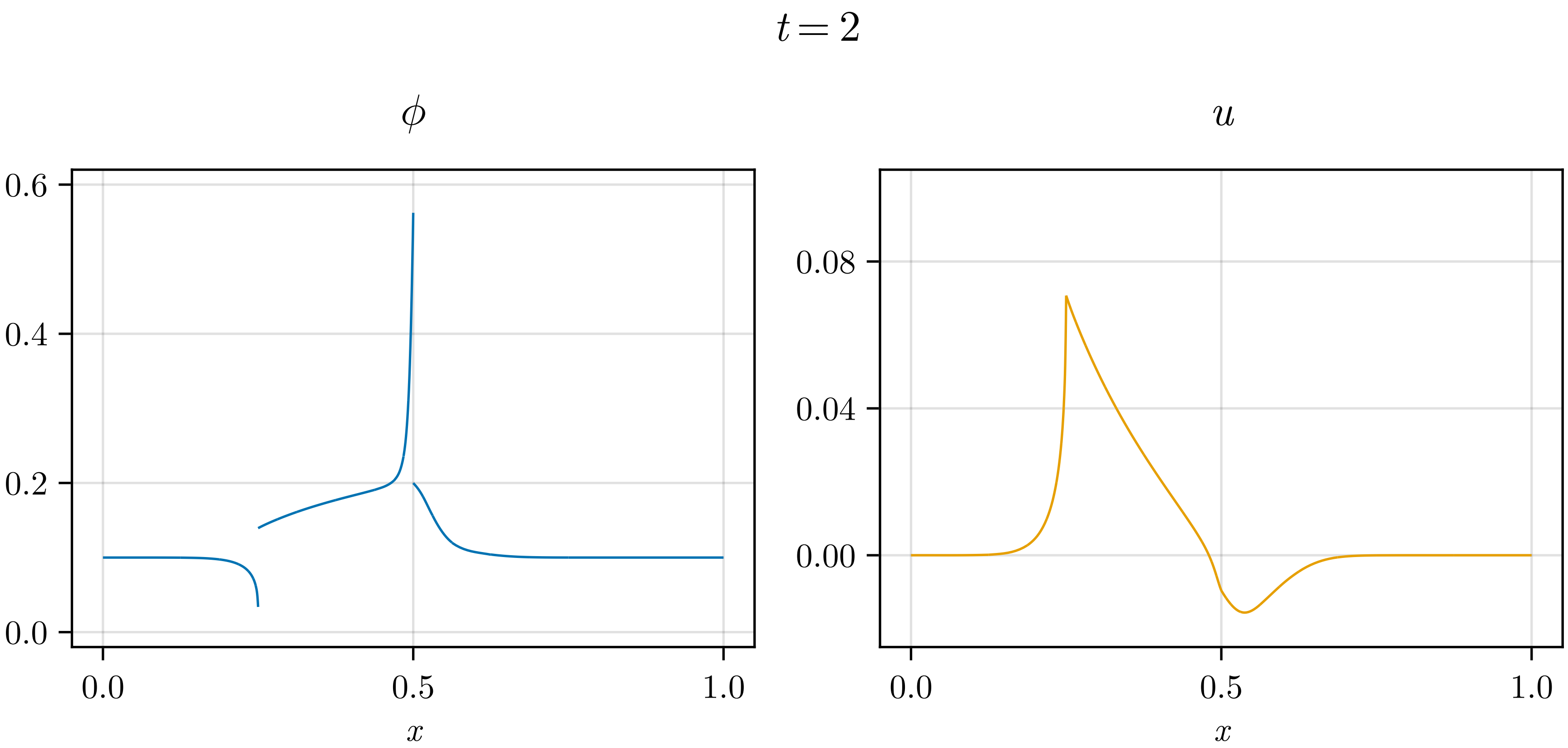

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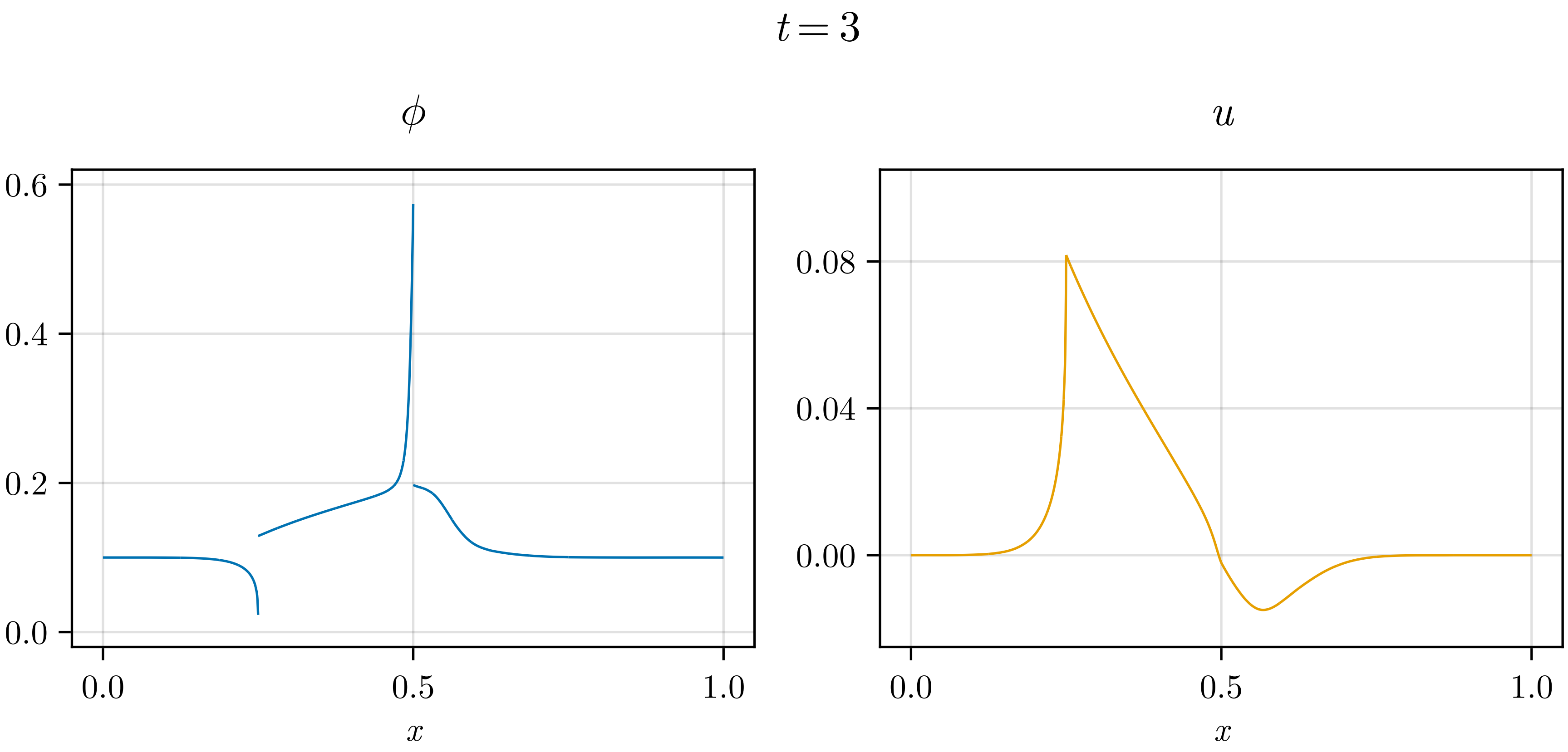

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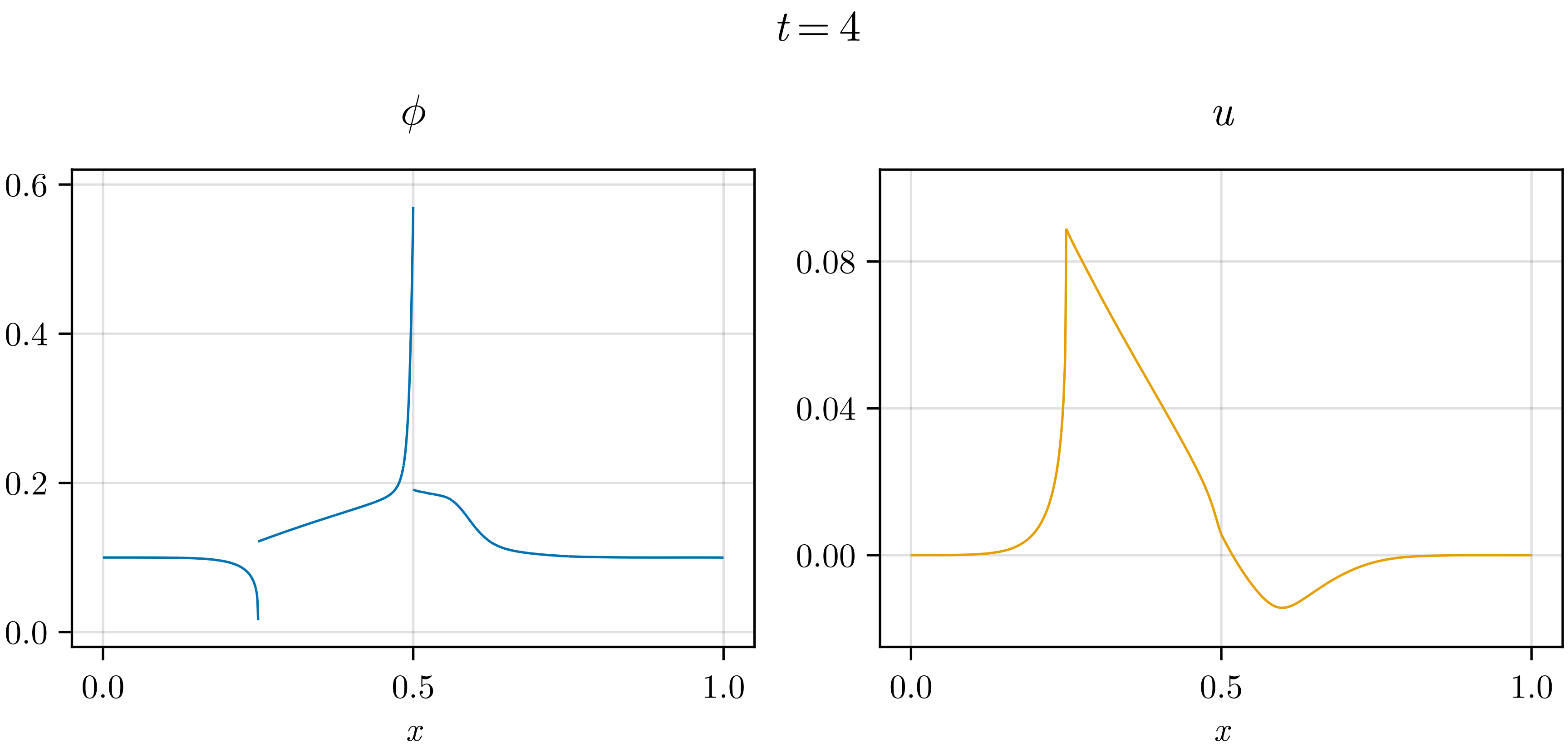

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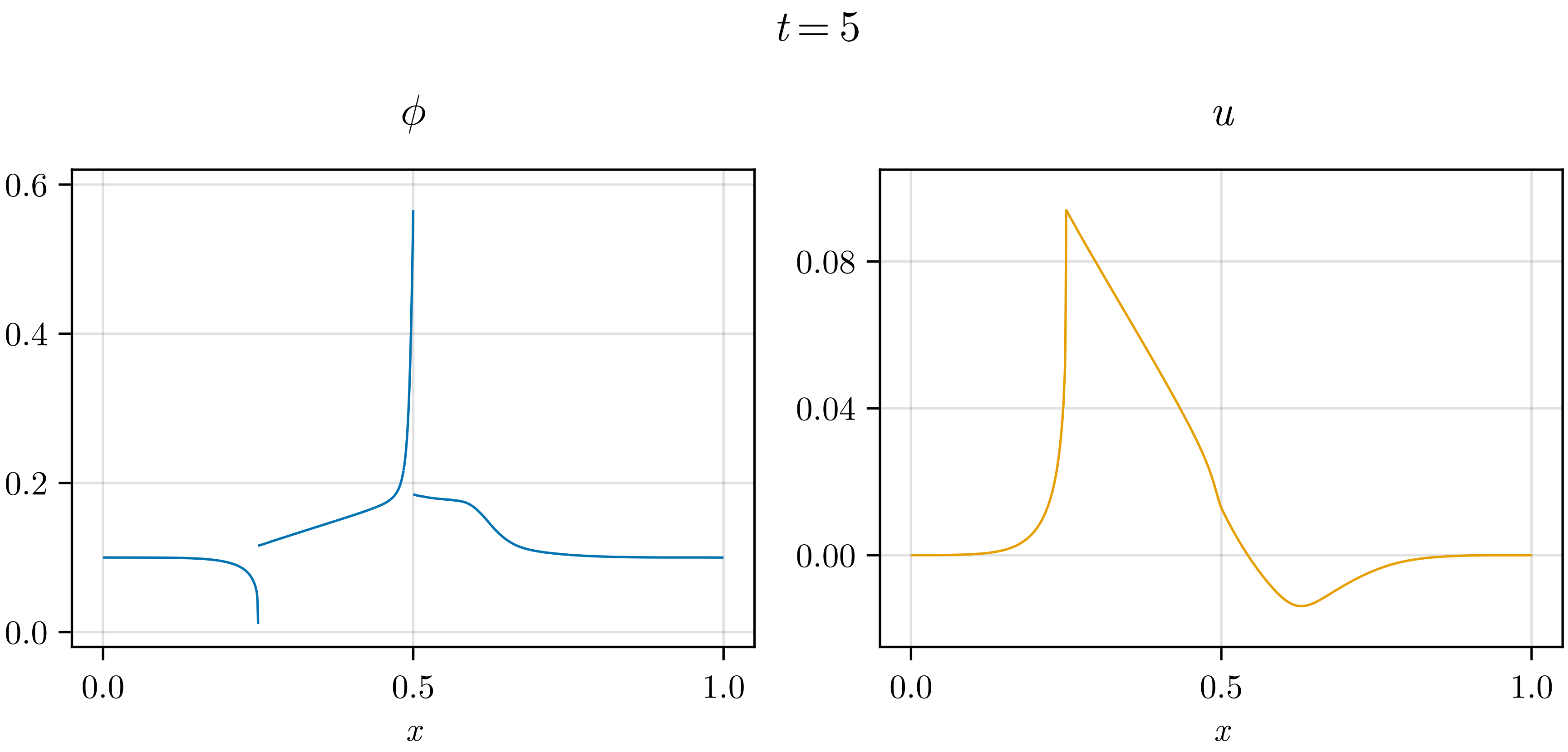

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- New general form of the problem (with either ϕ or λ): with locally Lipschitz functions α, β, ζ,

$$\partial_t \varphi = -\frac{\beta(\varphi)}{\sigma(u)} u - Q\partial_t u\,,$$

$$\partial_t u = \frac{1}{Q}\left(\nabla \cdot \alpha(\varphi)\,(\nabla u + \zeta(\varphi)) - \frac{\beta(\varphi)}{\sigma(u)} u \right).$$

Well-posedness in the viscous limit case

Setting $\kappa(v) := v/\sigma(v)$,

$$\begin{aligned} \partial_t \varphi &= -\beta(\varphi)\kappa(u)\,, \\ 0 &= \nabla \cdot \alpha(\varphi)(\nabla u + \zeta(\varphi)) - \beta(\varphi)\kappa(u)\,, \end{aligned}$$

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$\rightsquigarrow$ Picard iteration approach:

$$\varphi^{\text{new}}(t,\cdot) = \varphi_0 - \int_0^t \beta(\varphi^{\text{old}}(s,\cdot))\,\kappa(u[\varphi^{\text{old}}(s,\cdot)])\,\mathrm{d}s\,, \quad \text{for all } t \in [0,T]\,.$$

Well-posedness in the viscous limit case

Theorem (Bachmayr, B., Kreusser 2023)

Let $\varphi_0 \in L^\infty(\Omega)$ and $d = 1, 2$. Then for a $T > 0$, there exists a unique solution $(\varphi, u) \in C([0,T]; L^\infty(\Omega)) \times C([0,T]; H_0^1(\Omega))$.

Theorem (Bachmayr, B., Kreusser 2023)

Let $\varphi_0 \in C^{k,1}(\overline{\Omega})$, $k \in \mathbb{N}_0$. Then for a $T > 0$, there exists a unique solution $(\varphi, u) \in C([0,T]; C^{k,1}(\overline{\Omega})) \times C([0,T]; C^{k+1,\gamma}(\overline{\Omega}))$ for any $\gamma \in [0,1)$.

Existence and uniqueness for small $T \rightsquigarrow$ continuation up to maximal time of existence.

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Forward Euler argument yields:

Theorem (Bachmayr, B., Kreusser 2023)

Let $\varphi_0 \in \mathrm{BV}(\Omega)$. Then for a $T > 0$, there exists a solution $(\varphi, u) \in C([0,T]; \mathrm{BV}(\Omega)) \times C([0,T]; H_0^1(\Omega))$.

Well-posedness for the viscoelastic model

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As before: Given $\varphi \in L^\infty(\Omega_T) \rightsquigarrow$ there exists a unique solution $u[\varphi]$.

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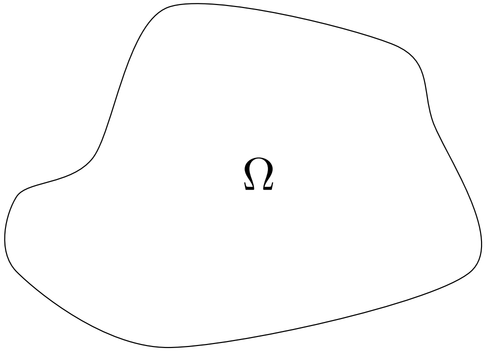
Theorem (Bachmayr, B., Kreusser 2023)

If $\varphi_0 \in C^{0,\alpha}(\overline{\Omega_i})$ and $u_0 \in C^{1,\alpha}(\overline{\Omega_i})$ for $i = 1, \dots, m$. Then for a $T > 0$, there exists a unique solution $(\varphi, u) \in C^{0,\alpha}_{\text{par}}(\overline{\Omega_i} \times [0,T]) \times C^{1,\alpha}_{\text{par}}(\overline{\Omega_i} \times [0,T])$ for $i = 1, \dots, m$.

(based on Dong, Xu 2021)



Well-posedness for the viscoelastic model

$$\partial_t \varphi = -\beta(\varphi)\kappa(u) - Q\partial_t u\,, \qquad \varphi(0,\cdot) = \varphi_0\,,$$

$$\partial_t u = \frac{1}{Q}\Big(\nabla\cdot\alpha(\varphi)(\nabla u + \zeta(\varphi)) - \beta(\varphi)\kappa(u)\Big)\,,$$

$+$ sufficiently smooth initial and boundary data for u.

As before: Given $\varphi \in L^\infty(\Omega_T) \rightsquigarrow$ there exists a unique solution $u[\varphi]$.

Notion of solution $\rightsquigarrow$ picard iteration approach as before:

$$\varphi^{\text{new}}(t,\cdot) = \varphi_0 - Q(u[\varphi^{\text{old}}](t,\cdot) - u_0) - \int_0^t \beta(\varphi^{\text{old}}(s,\cdot))\kappa(u[\varphi^{\text{old}}](s,\cdot))\,\mathrm{d}s\,.$$

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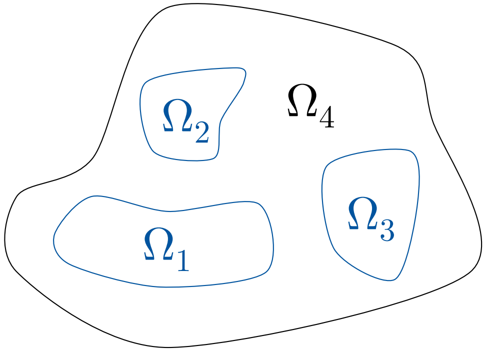

Numerical method

Solve parabolic equation for fixed φ

$$\partial_t u = \frac{1}{Q}\left(\nabla_x \cdot \alpha(\varphi)(\nabla_x u + \zeta(\varphi)) - \beta(\varphi)\,\frac{u}{\sigma(u)}\right), \quad u(0,\cdot) = u_0\,,$$

Numerical method

Solve parabolic equation for fixed φ

$$\partial_t u^{\text{new}} = \frac{1}{Q}\left(\nabla_x \cdot \alpha(\varphi)(\nabla_x u^{\text{new}} + \zeta(\varphi)) - \beta(\varphi)\,\frac{u^{\text{new}}}{\sigma(u^{\text{old}})}\right), \qquad u^{\text{new}}(0,\cdot) = u_0\,,$$

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Solve parabolic equation for fixed φ

$$\partial_t u = \nabla_x \cdot \tilde{\alpha}(\varphi)(\nabla_x u + \zeta(\varphi)) - \tilde{\beta}(\varphi)\frac{u}{\overline{\sigma}}\,, \qquad u(0,\cdot) = u_0\,,$$

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Solve parabolic equation for fixed φ

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using an adaptive space-time least-squares method for the linear subproblems.

(Führer, Karkulik 2021; Gantner, Stevenson 2021; 2024)

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Idea: Consider equivalent system

$$G(u, \eta) := \begin{pmatrix} \operatorname{div}(u, \eta) + \tilde{\beta}(\varphi)\frac{u}{\overline{\sigma}} \\ \eta + \tilde{\alpha}(\varphi)\, \nabla_x u \\ u(0, \cdot) \end{pmatrix} = \begin{pmatrix} 0 \\ -\tilde{\alpha}(\varphi)\zeta(\varphi) \\ u_0 \end{pmatrix} =: R ,$$

Numerical method

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and compute

$$(u_\delta, \eta_\delta) = \operatorname*{arg\,min}_{(v_\delta,\mu_\delta)\in U_\delta} \|G(v_\delta,\mu_\delta) - R\|_{L^2(\Omega_T)\times L^2(\Omega_T;\mathbb{R}^d)\times L^2(\Omega)},$$

for a suitably chosen subspace $U_\delta \subset U := (L^2(0,T;H^1(\Omega)) \times L^2(\Omega_T;\mathbb{R}^d)) \cap H_{\mathrm{div}}(\Omega_T)$.

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Residual is a reliable nonlinear error estimator $\rightsquigarrow$ adaptive refinement & error control.

Numerical method

Motivation: fixed point iteration for mild solution of φ:

$$\varphi^{\mathsf{new}}(t,\cdot) = \varphi_0 + Q(u[\varphi^{\mathsf{old}}](t,\cdot) - u_0) - \int_0^t \beta(\varphi^{\mathsf{old}}(s,\cdot))\,\kappa(u[\varphi^{\mathsf{old}}](s,\cdot))\,\mathrm{d}s\,.$$

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Discretization by space-time polynomial ansatz:

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FOSLS solver

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FOSLS solver

Can prove convergence of this approach if we assume $\|\nabla_x u[\varphi]\|_{L^\infty(\Omega_T)} \leq C < \infty$.

Proof sketch:

- Show Lipschitz-estimate for the parabolic solution operator w.r.t. $\|\cdot\|_{L^2(\Omega_T)}$,
- Perturbed fixed-point iteration results yield convergence of the discrete iteration,
- For time slices a slightly adapted norm is needed to control terminal errors.

(Bachmayr, B. 2024)

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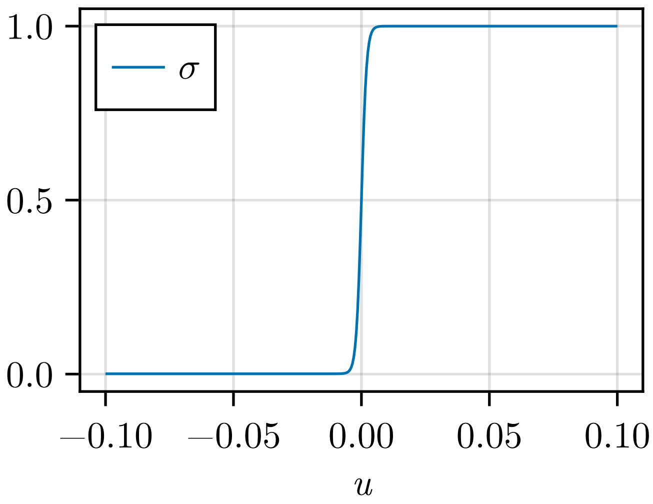
Adaptive space-time method ⇝ local time-steps.

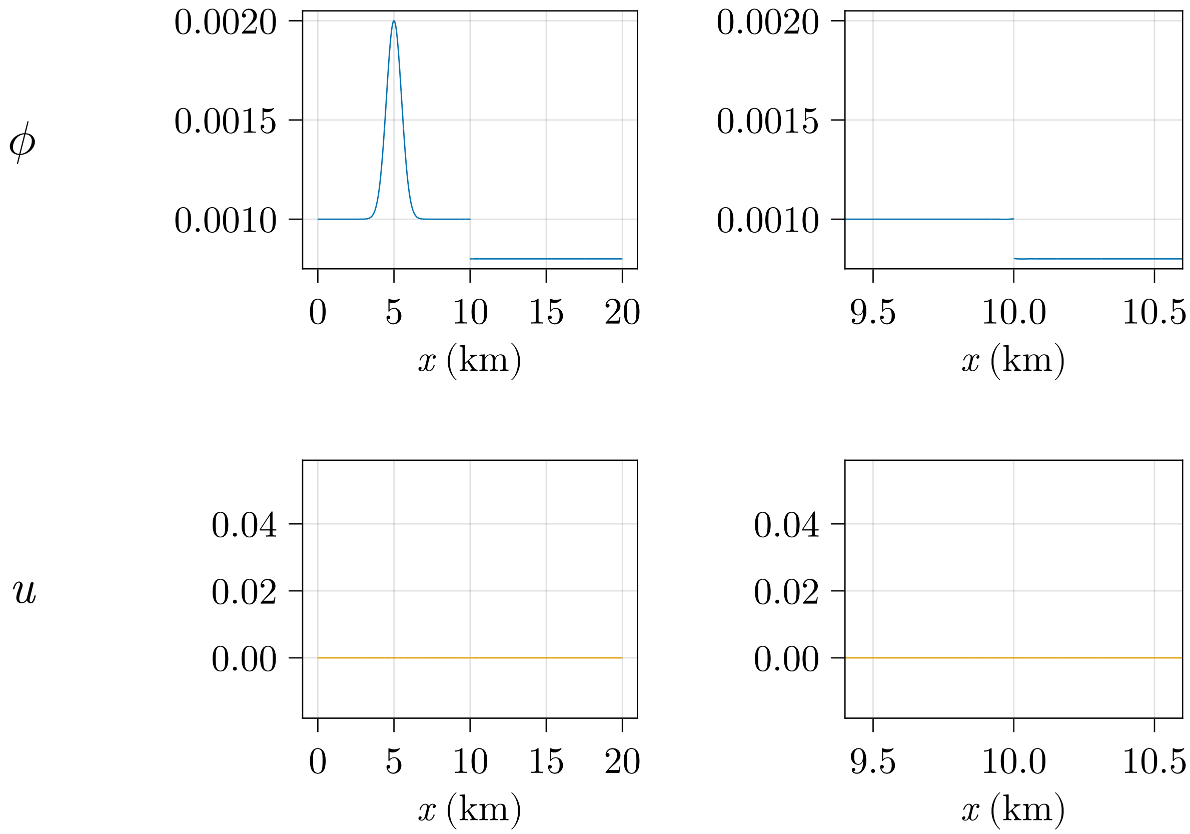
Numerical tests

“Realistic” parameter choice:

$$\begin{aligned}
\Omega &= (0\,,20)\ \text{km}\\
T &= 1.5\ \text{Myr}\\
\alpha(\varphi) &= 1000\,(1-\mathsf{exp}(-\varphi))^3\\
\beta(\varphi) &= (1-\mathsf{exp}(-\varphi))^2\\
\zeta(\varphi) &= \mathsf{exp}(-\varphi)\\
\sigma(u) &= \frac{10^{-3}+\mathsf{exp}(10^3\,u)}{1+\mathsf{exp}(10^3\,u)}\\
Q &= 1/60
\end{aligned}$$



Numerical tests



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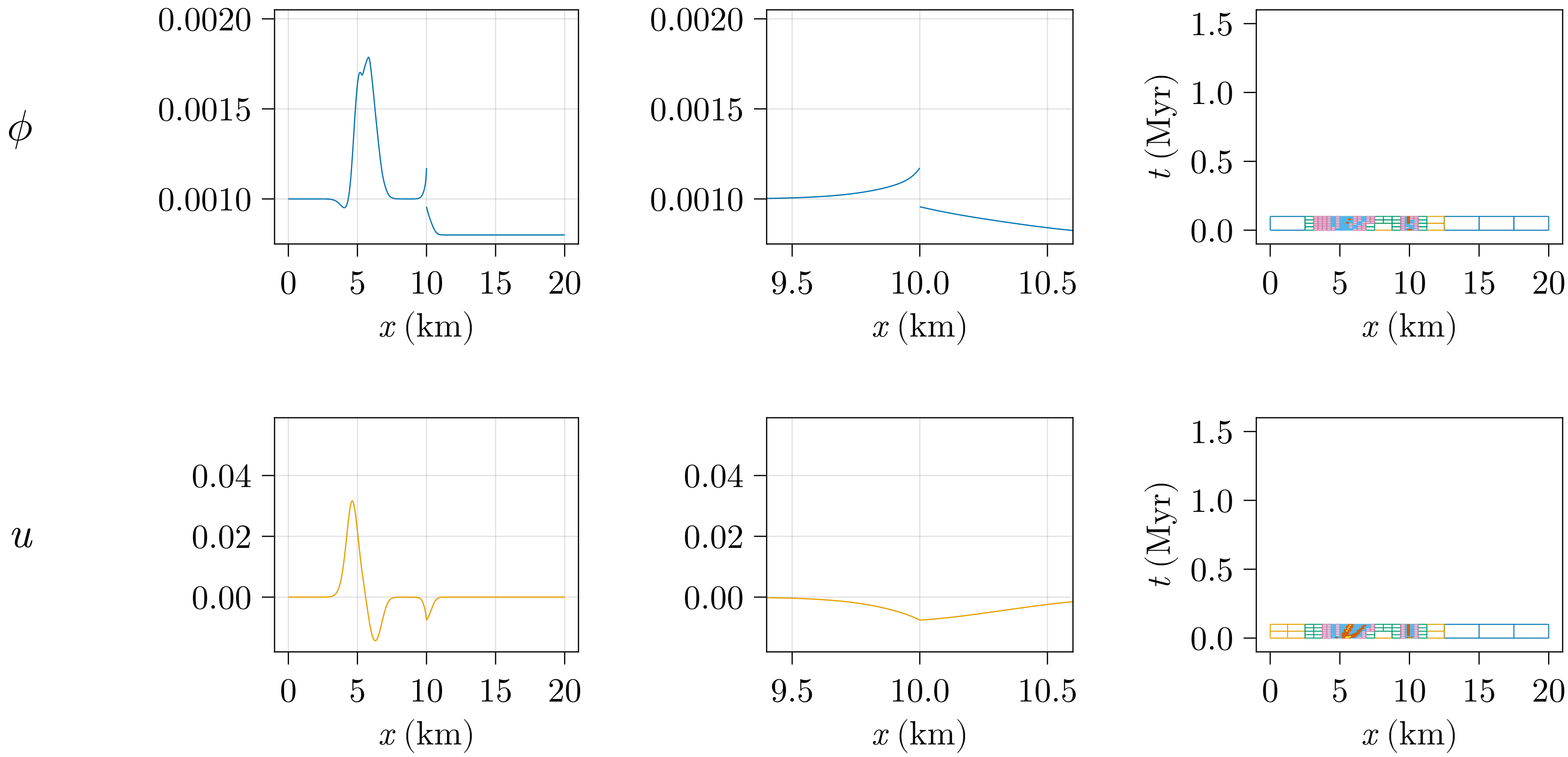

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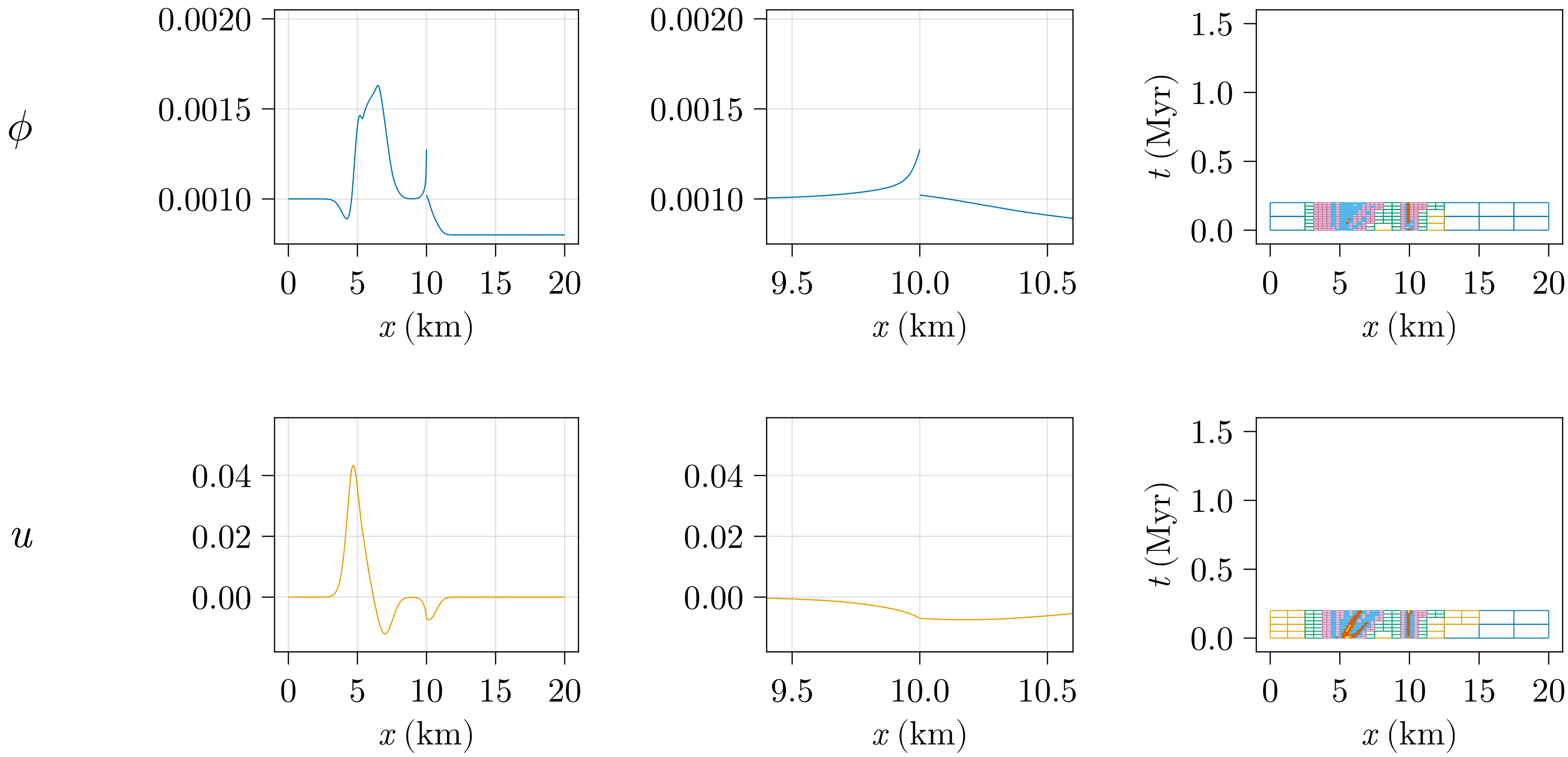

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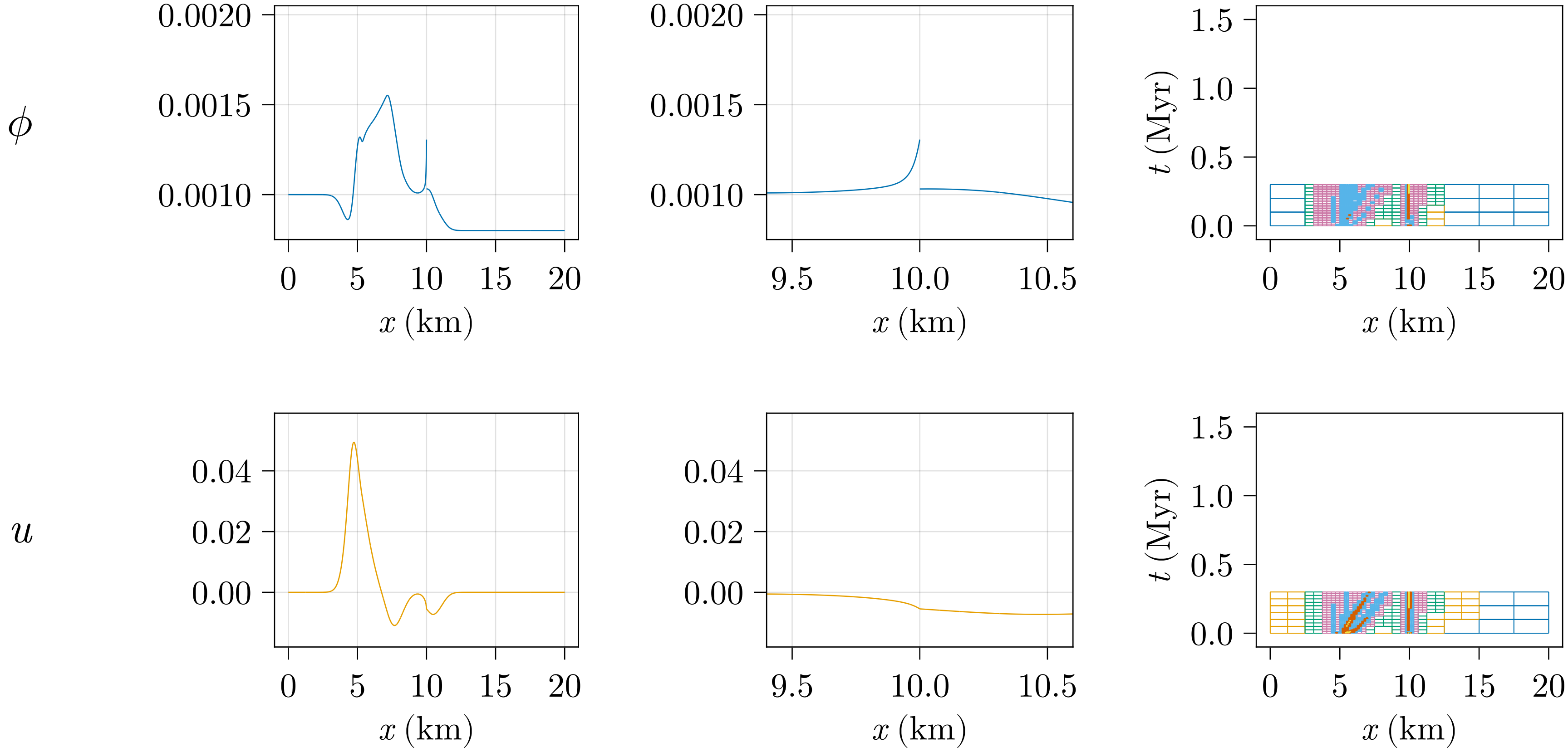

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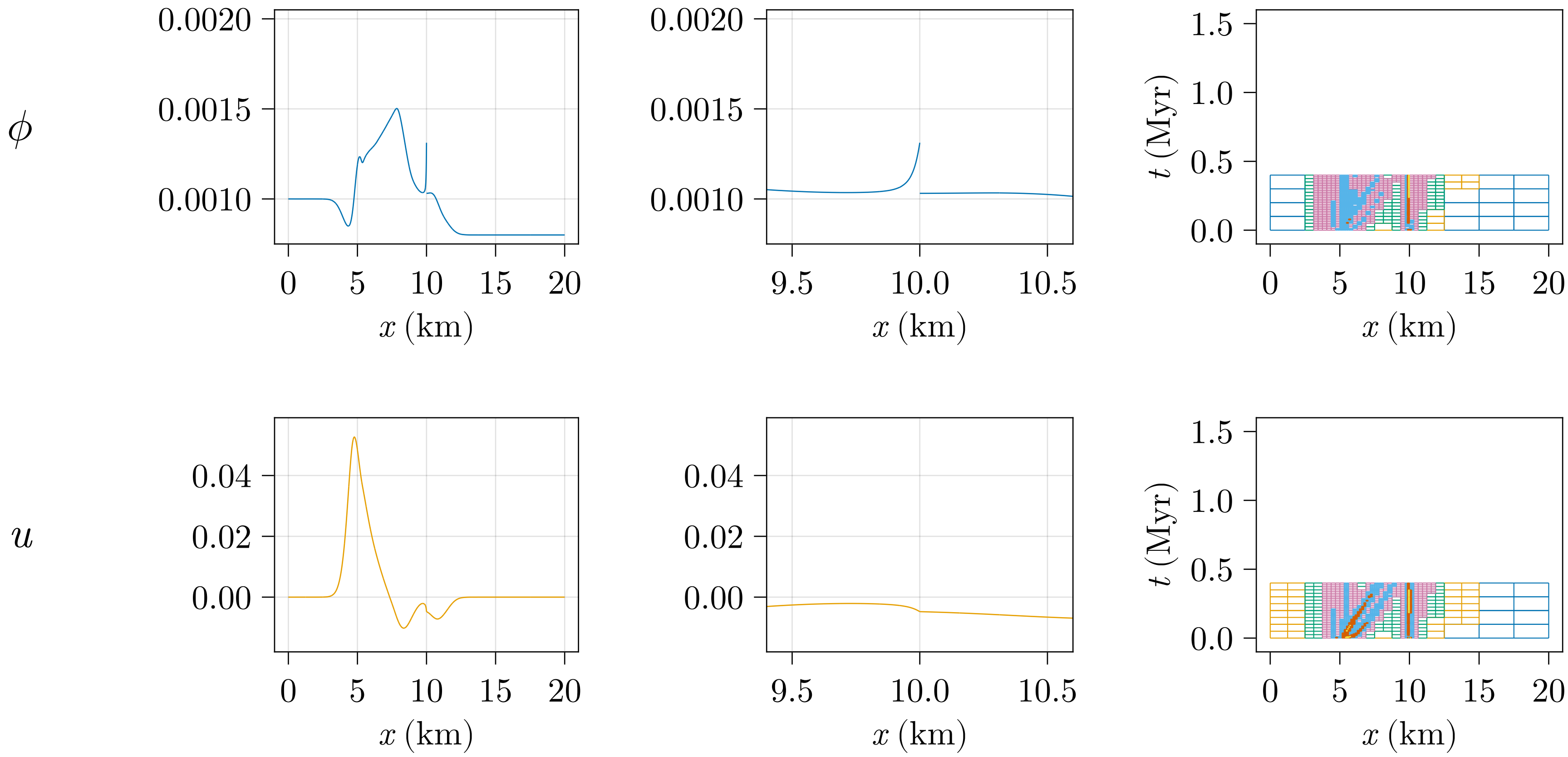

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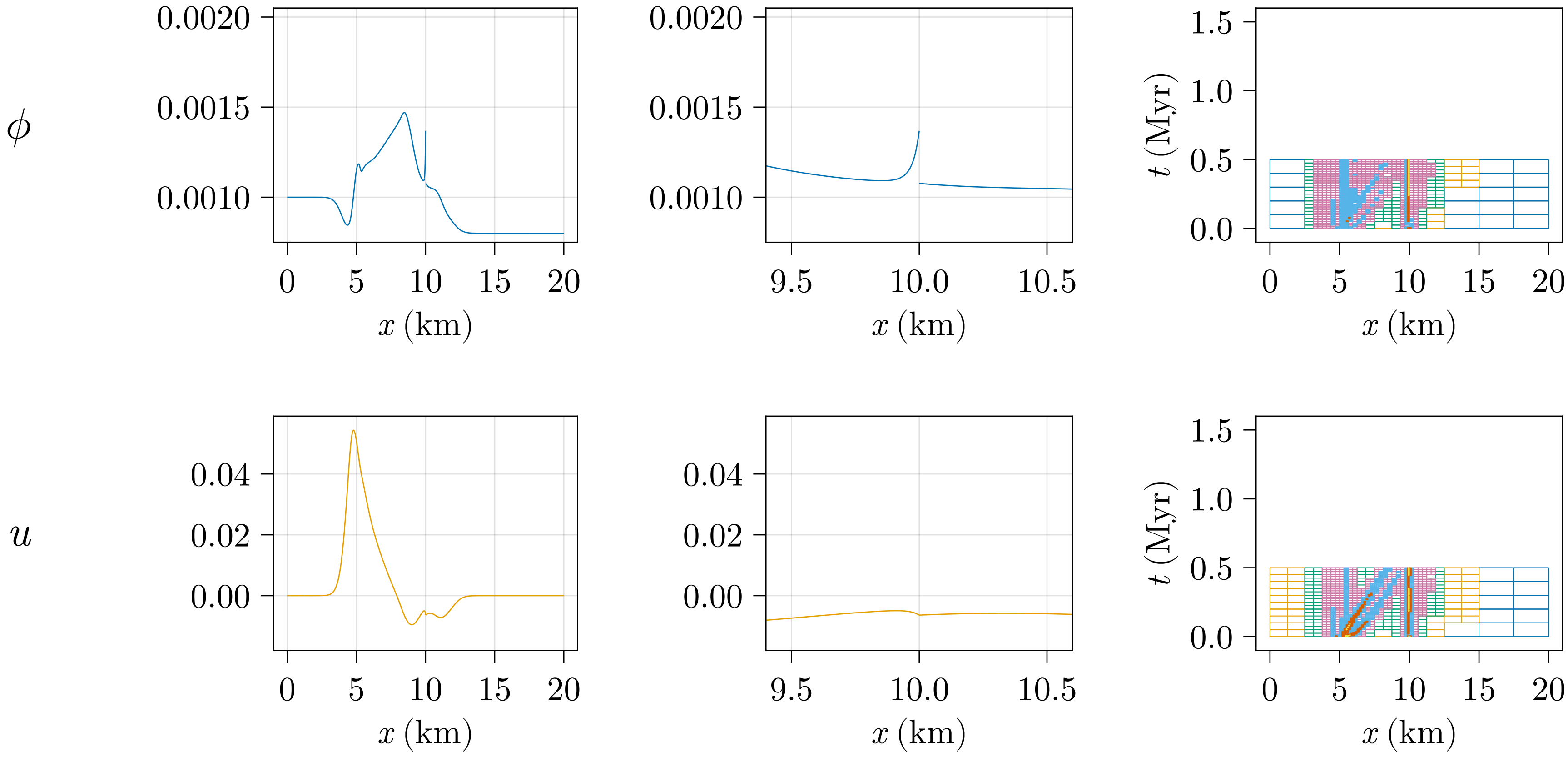

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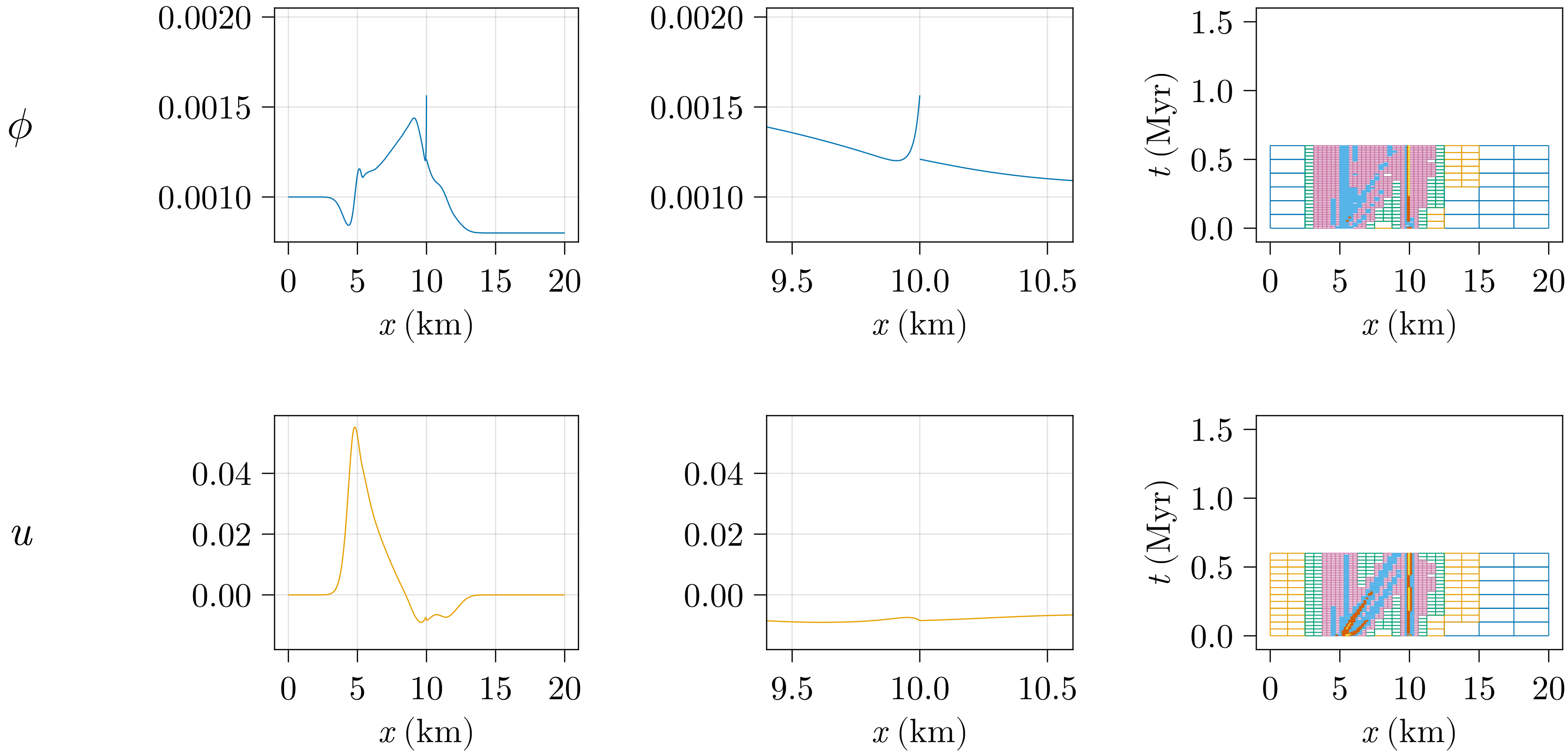

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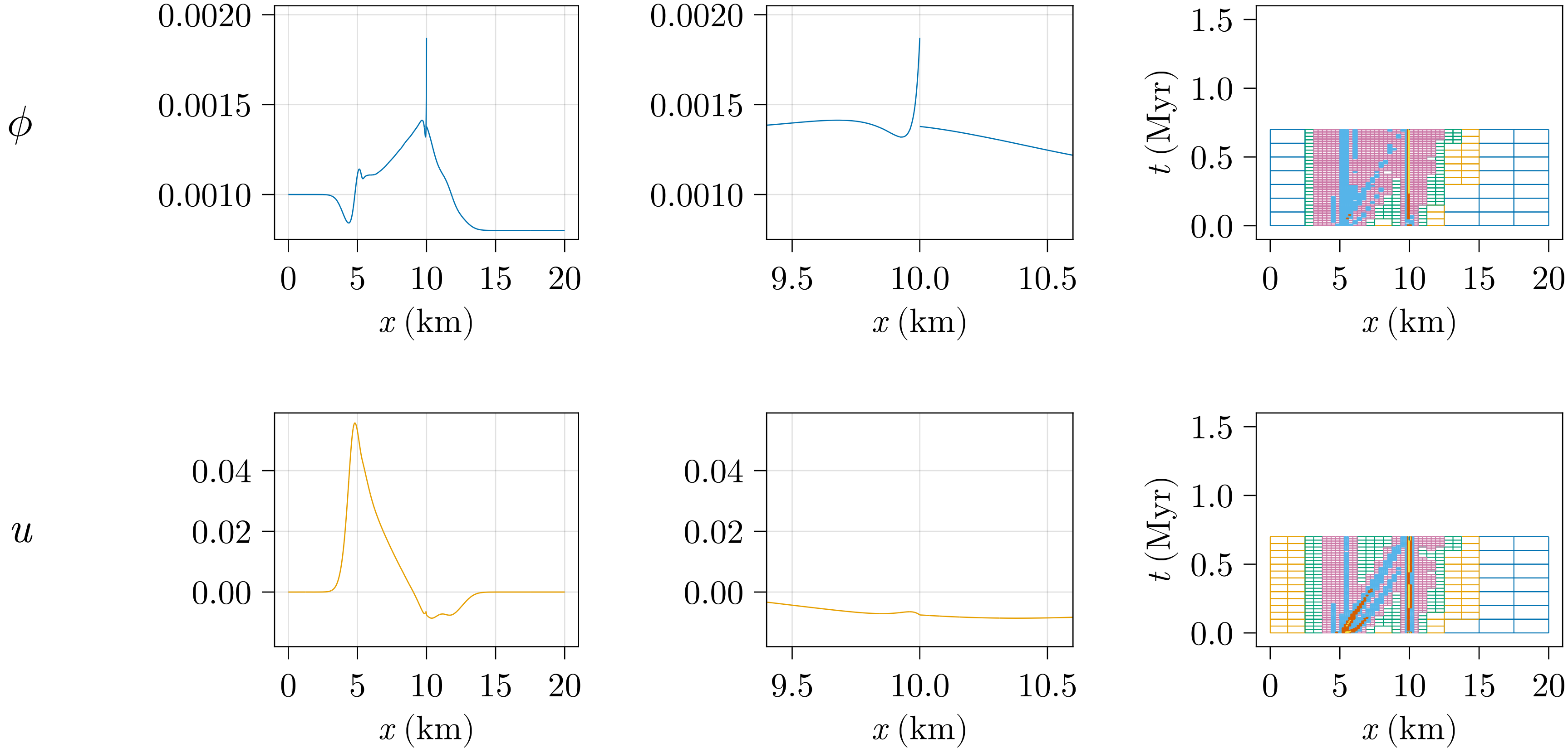

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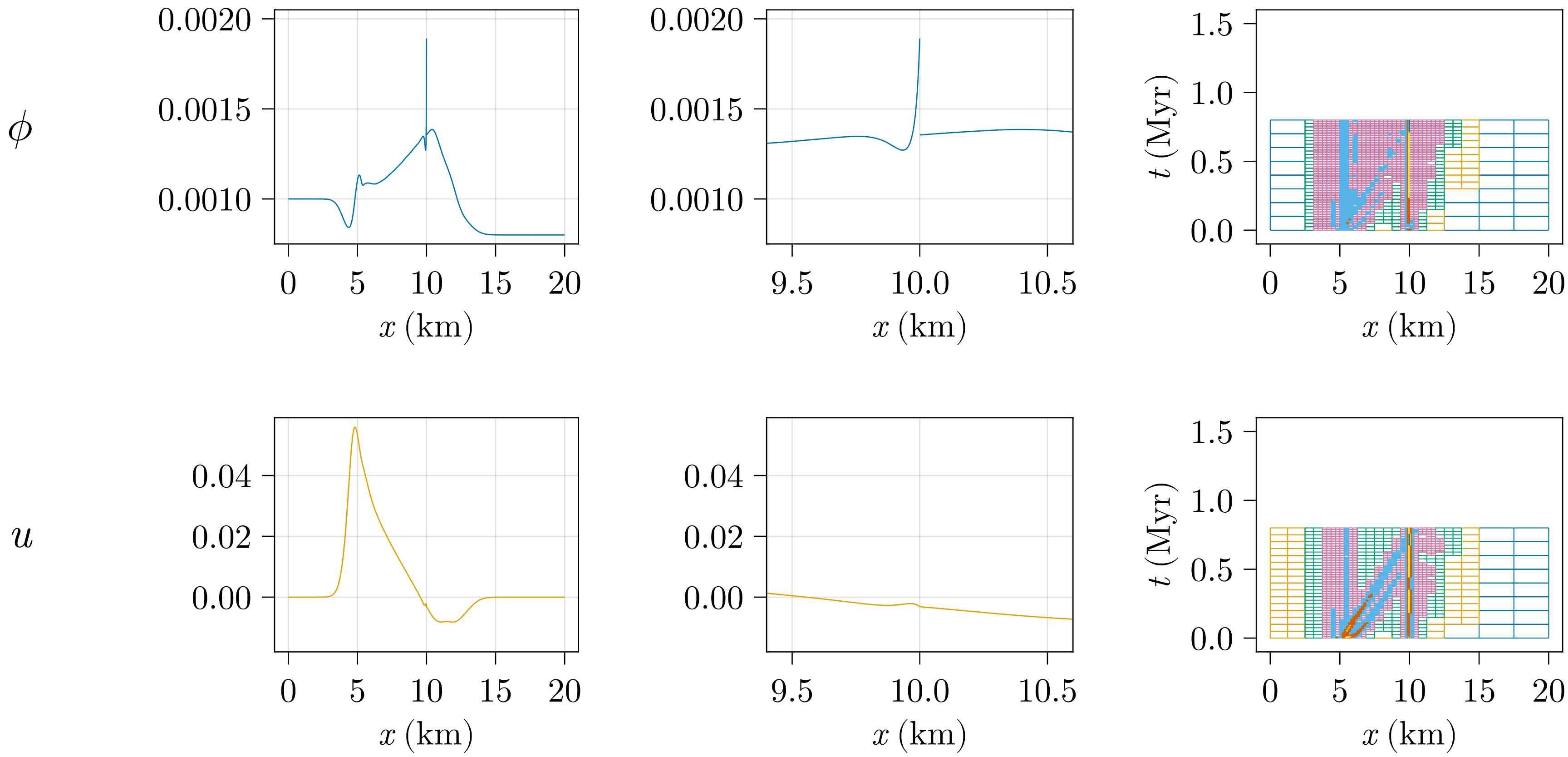

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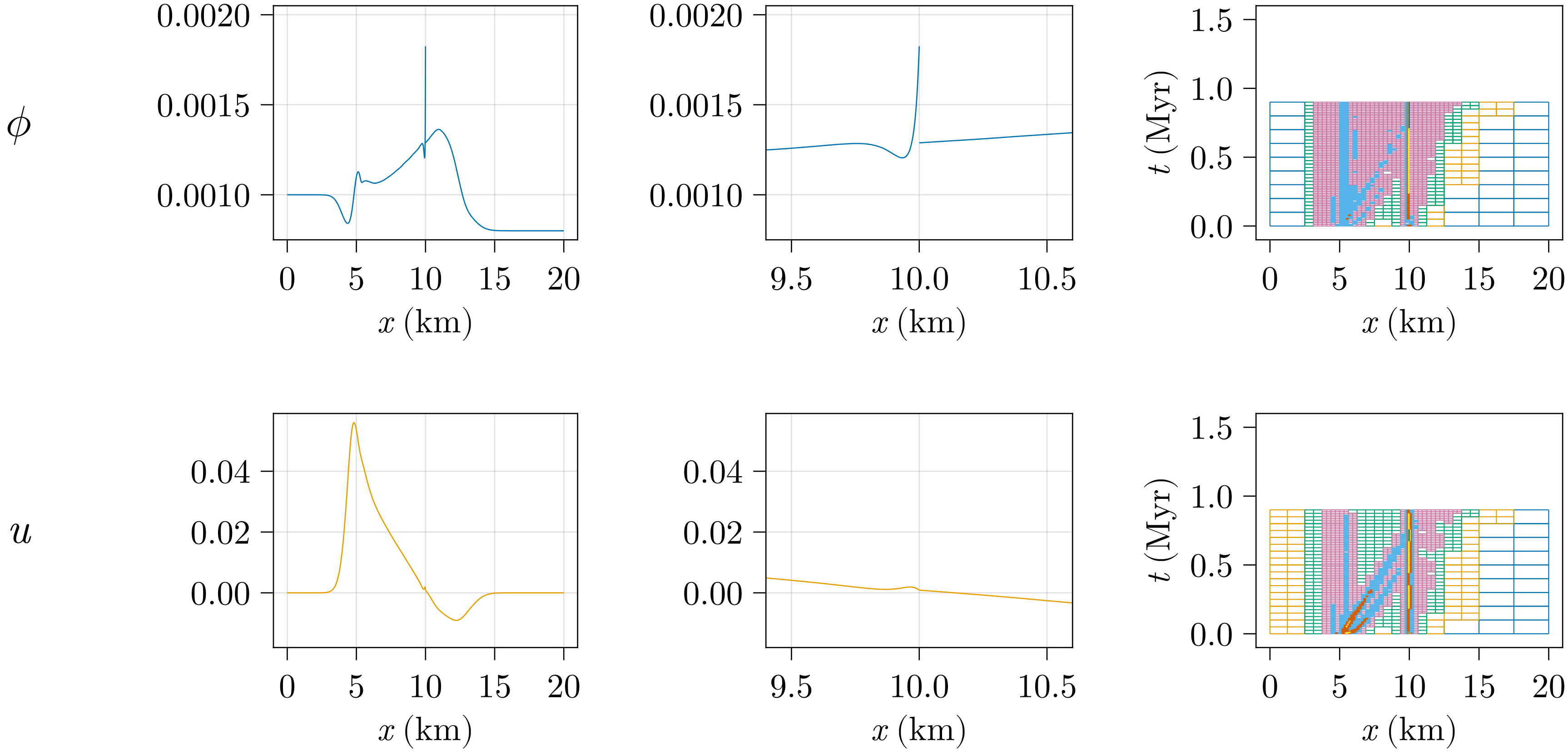

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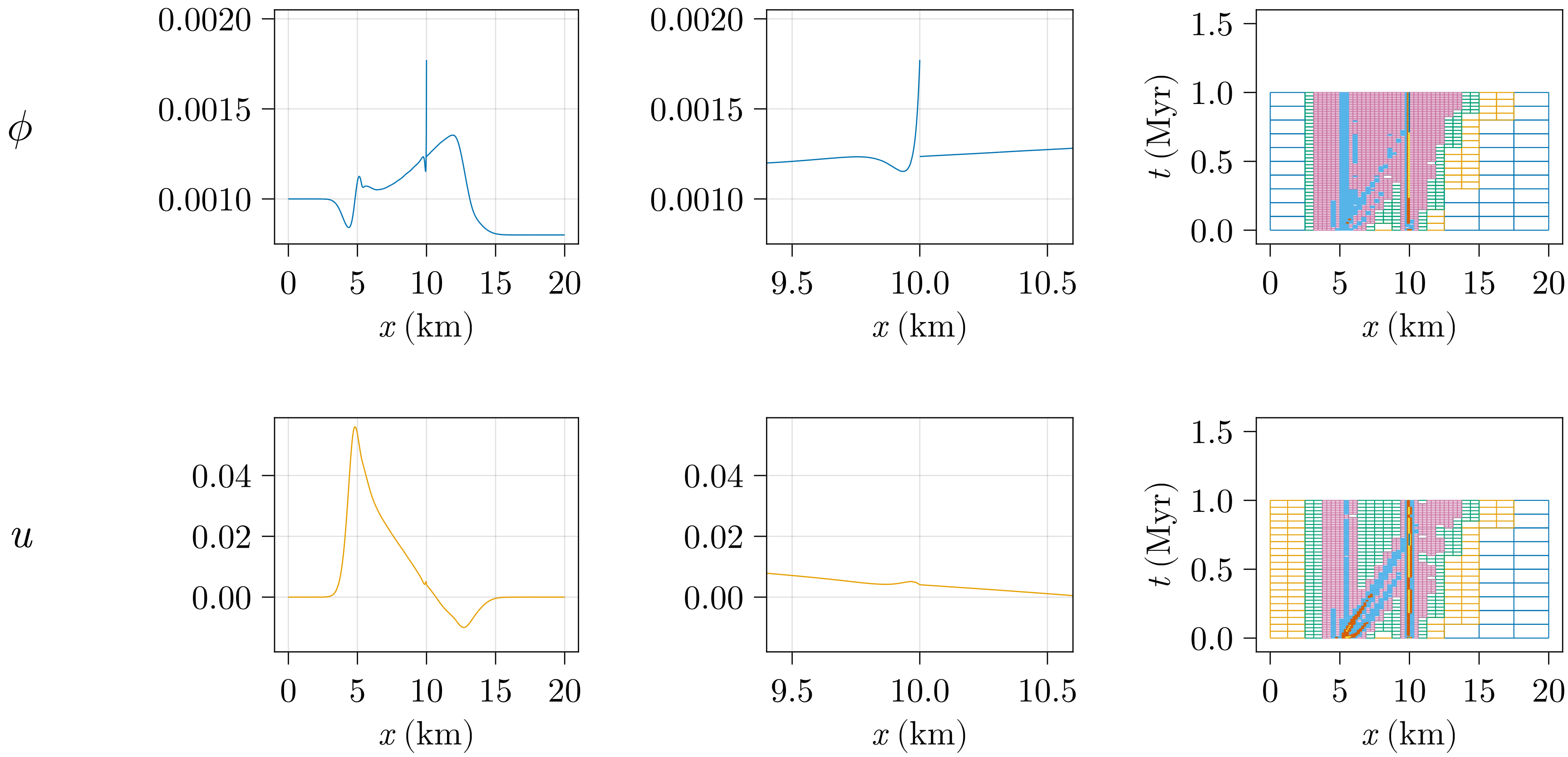

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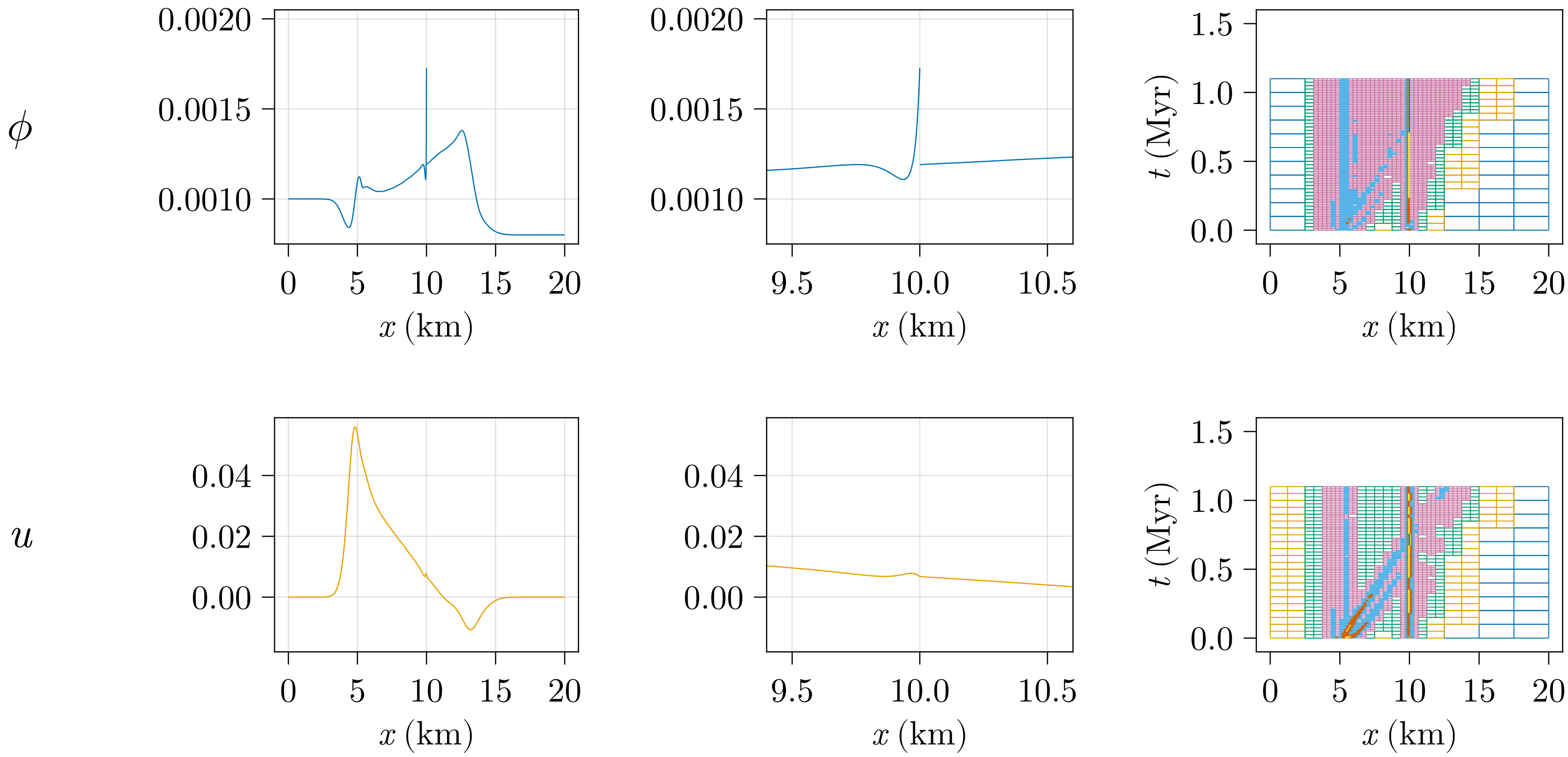

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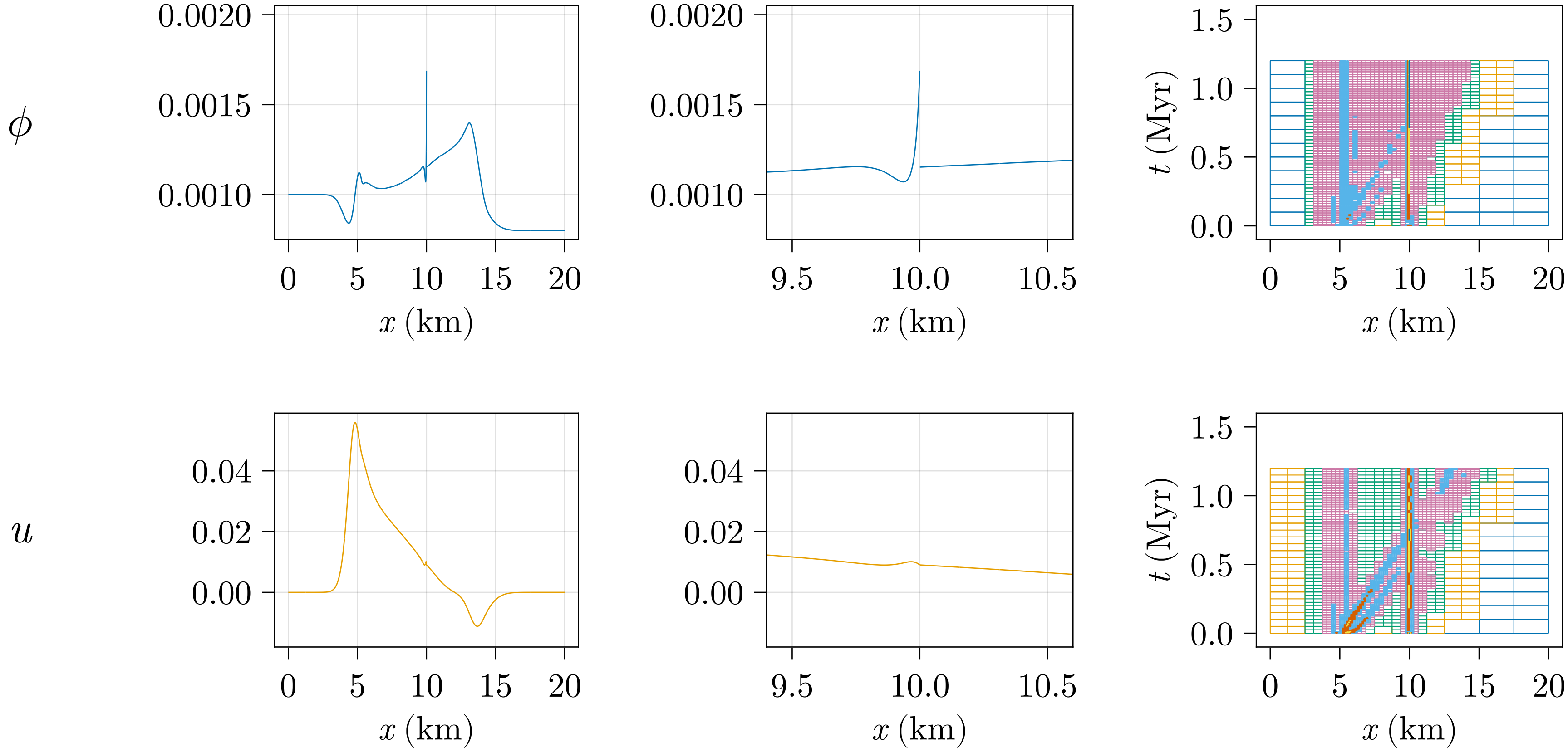

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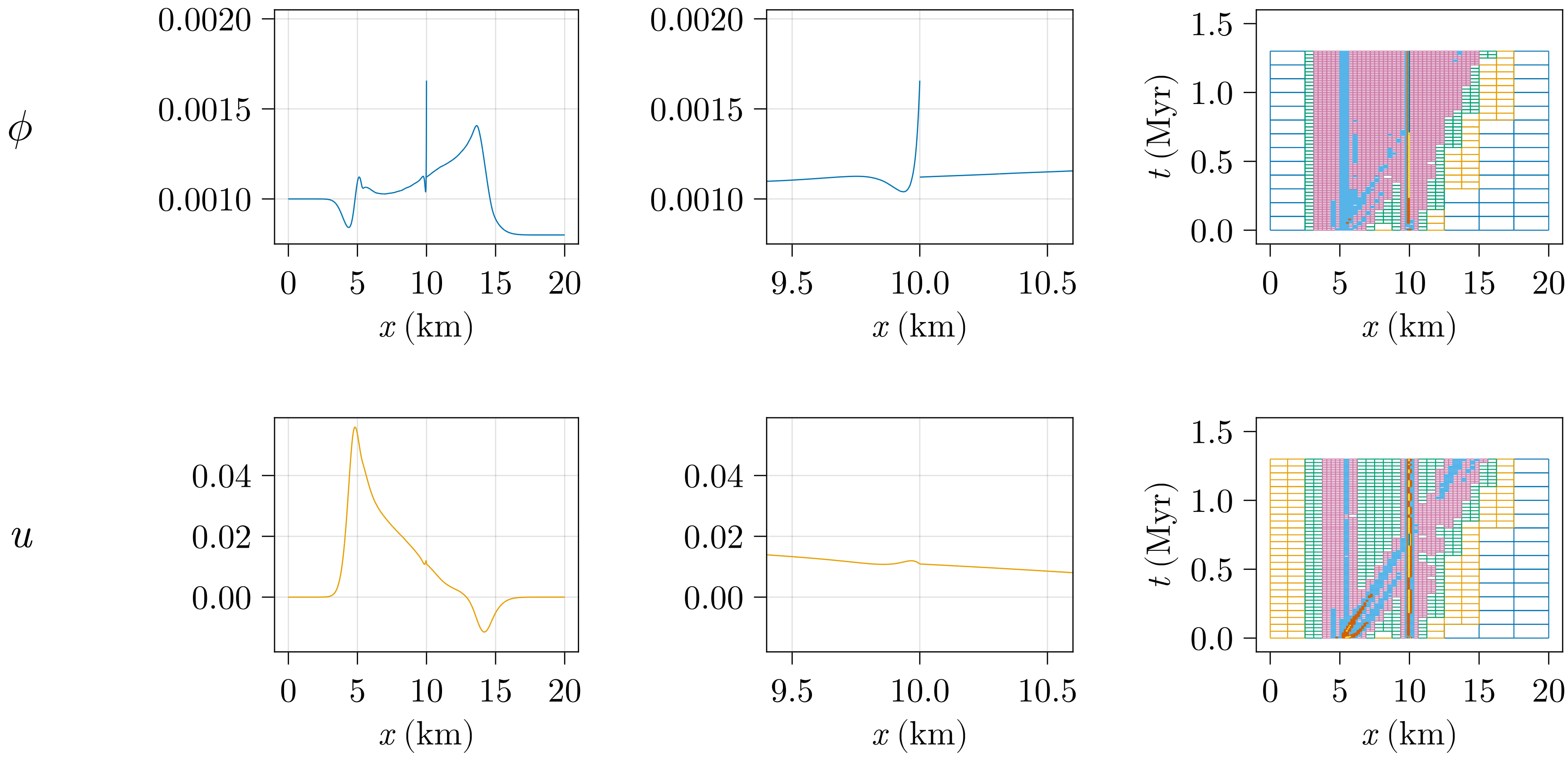

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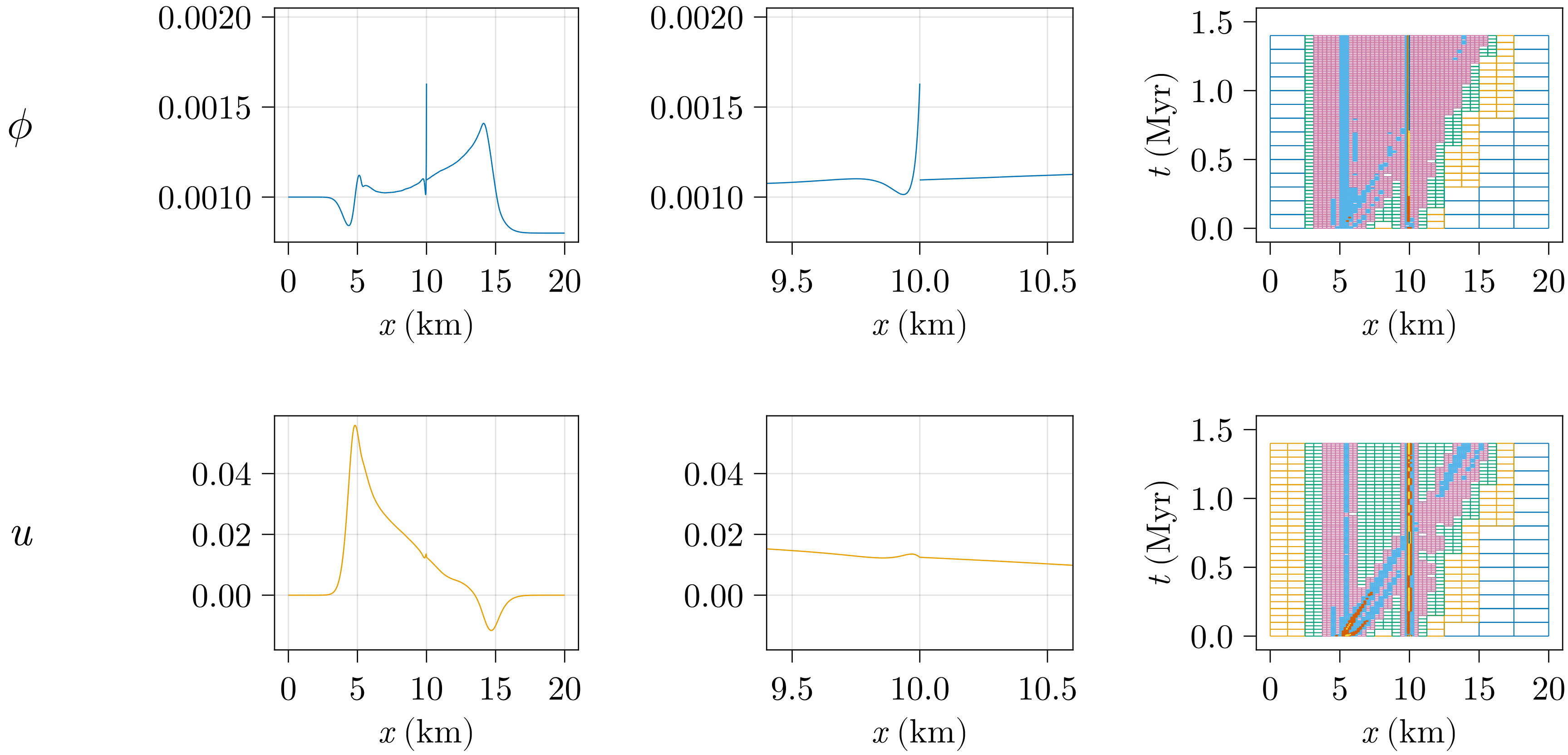

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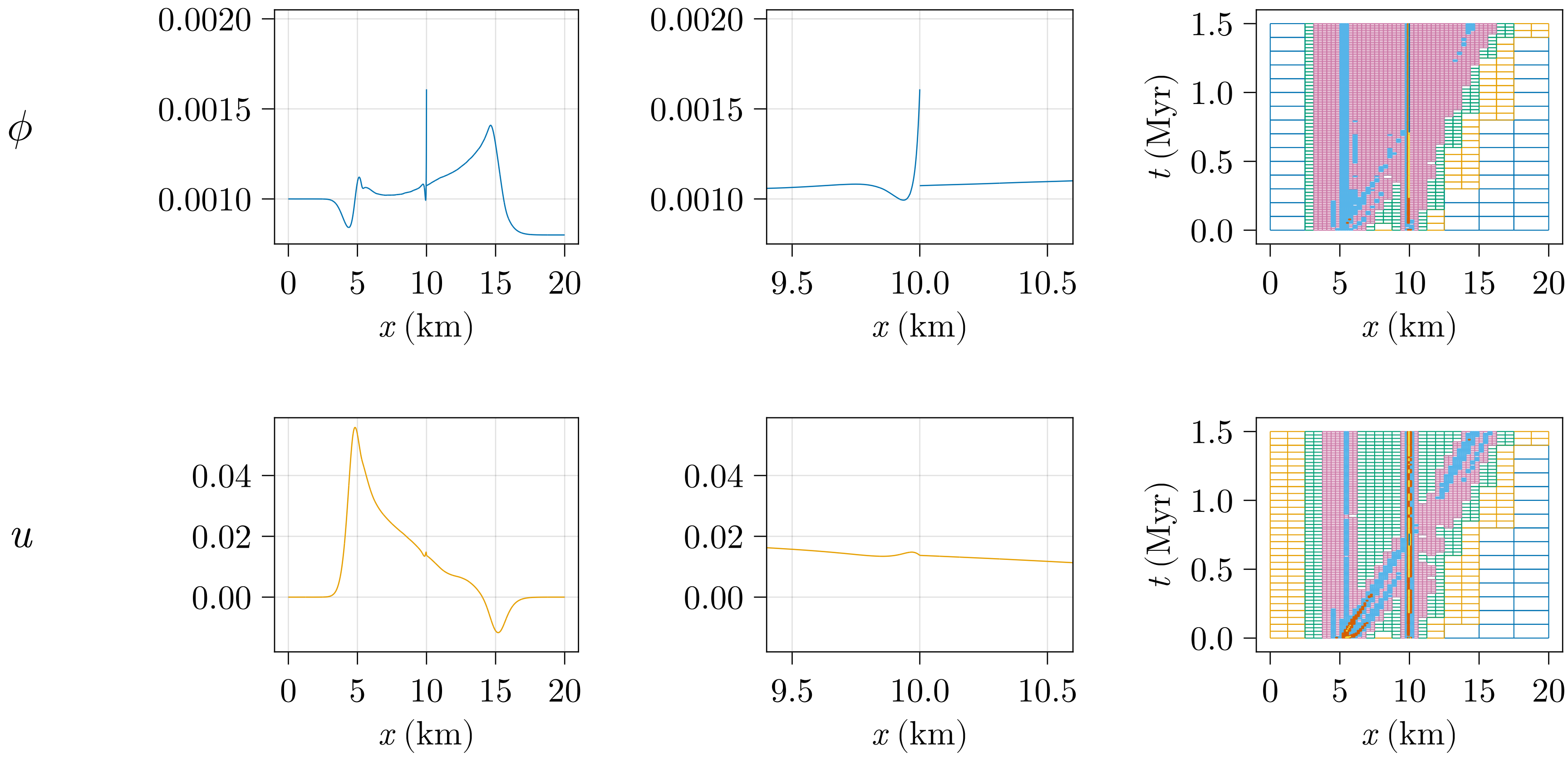
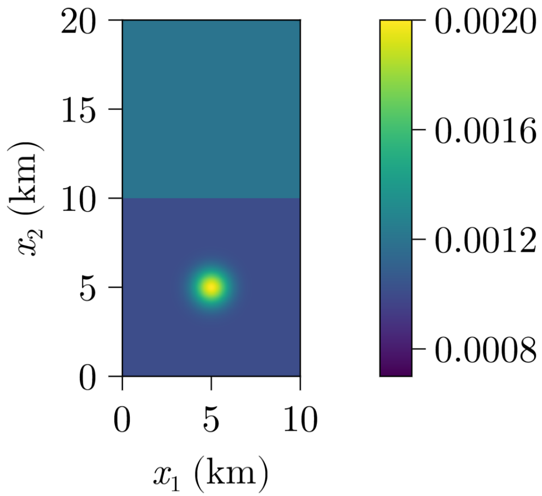


Numerical tests

ϕ



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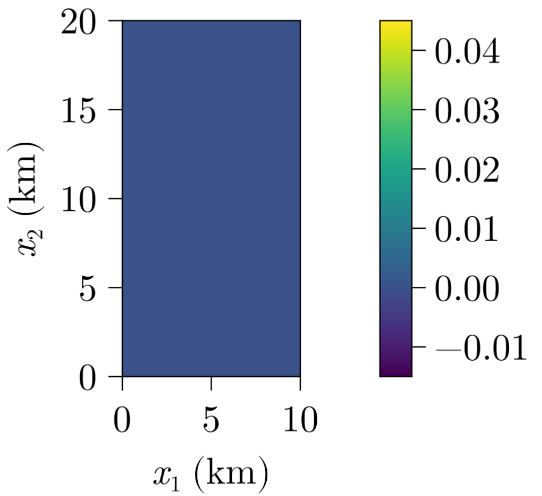

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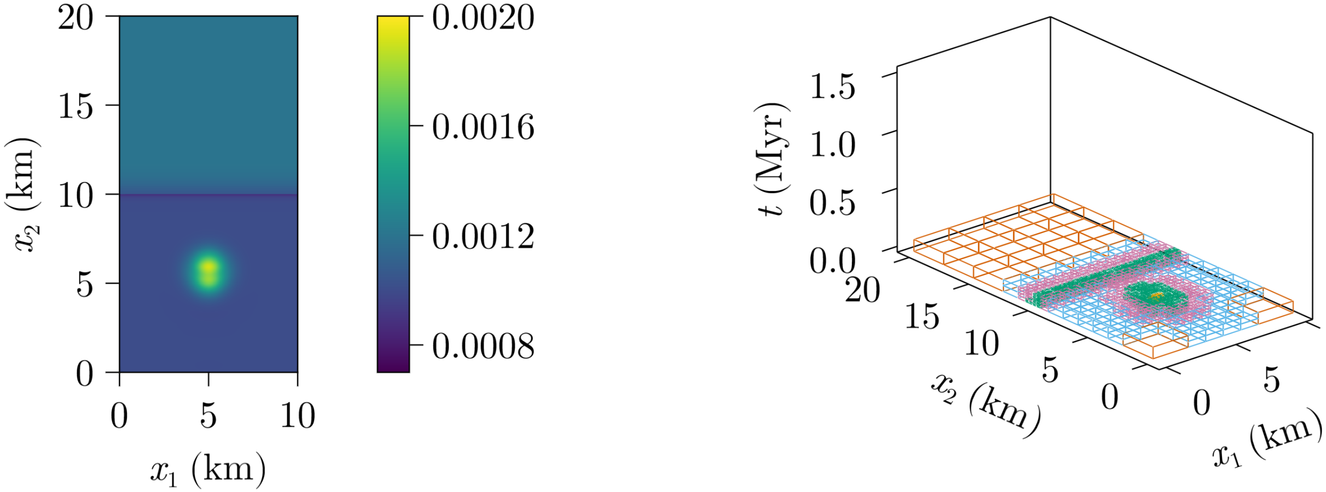

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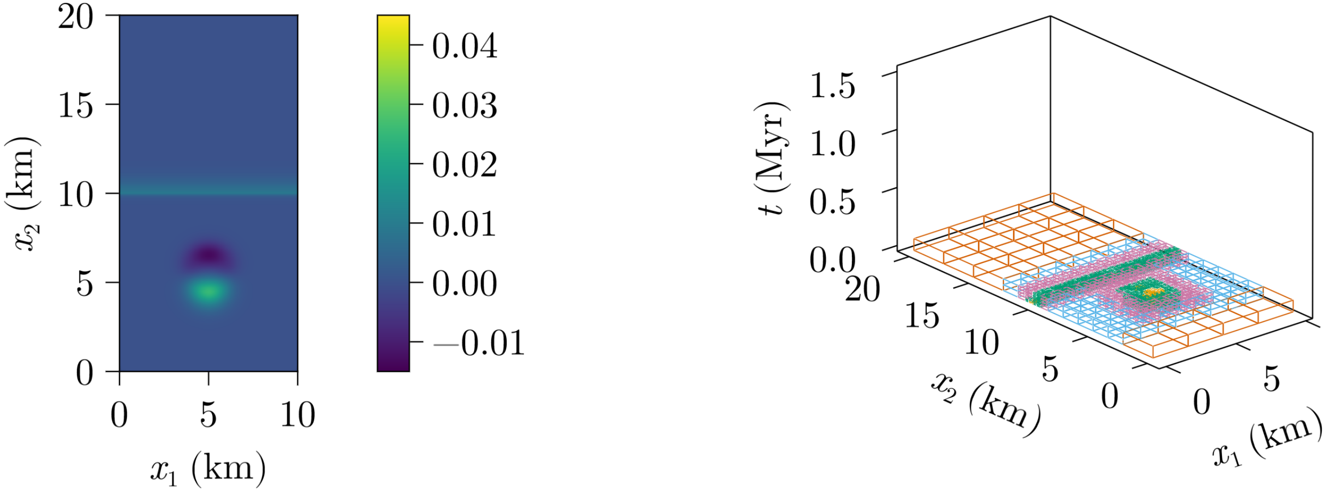

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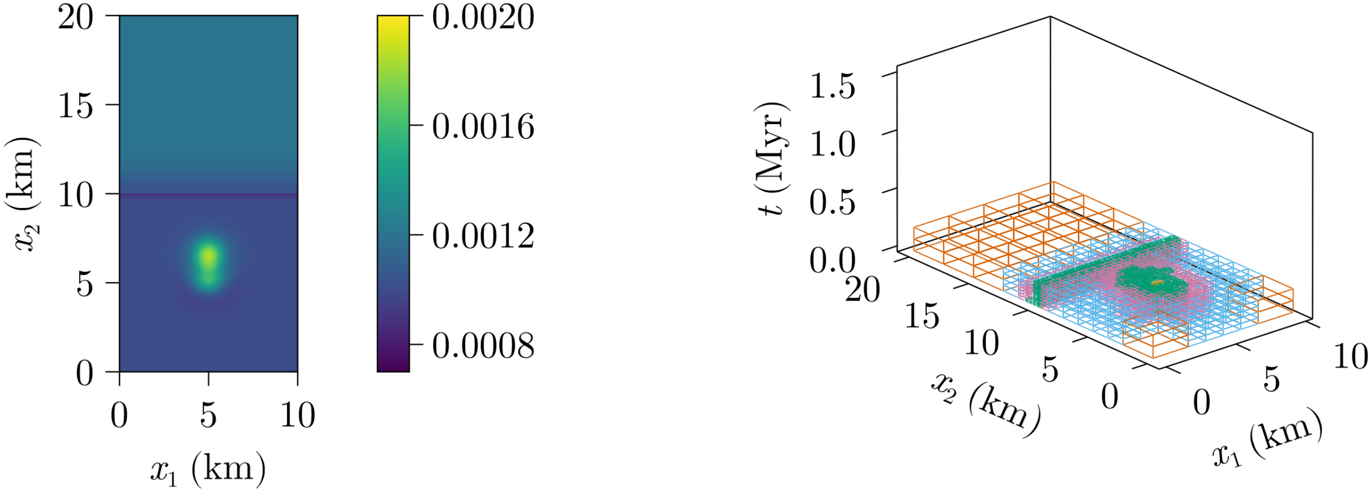

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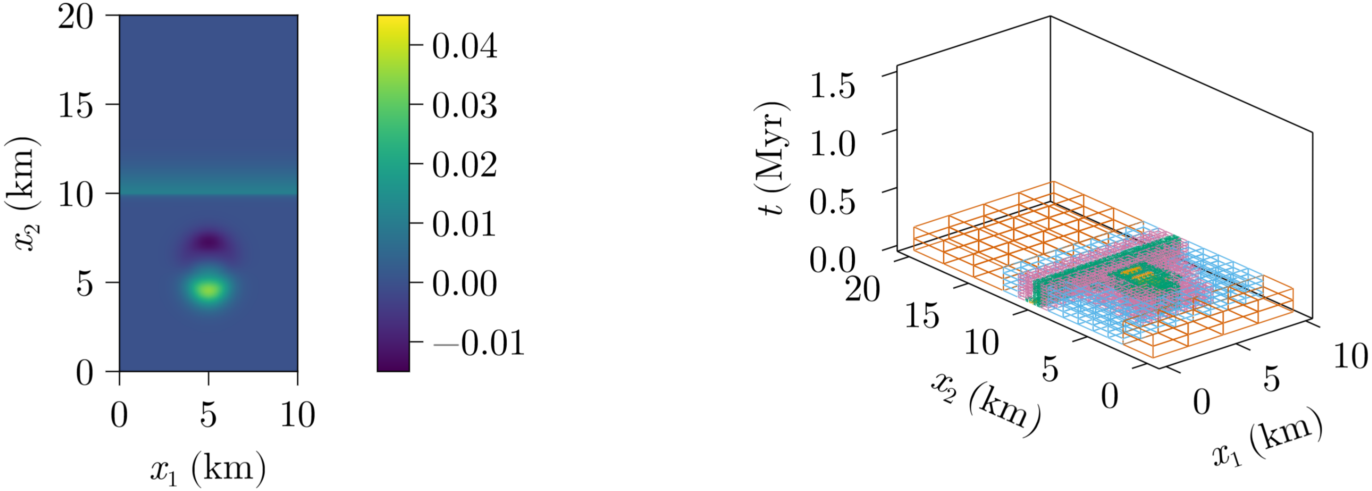

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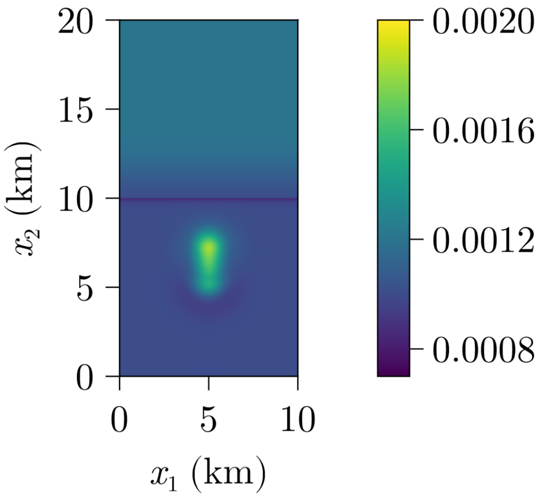

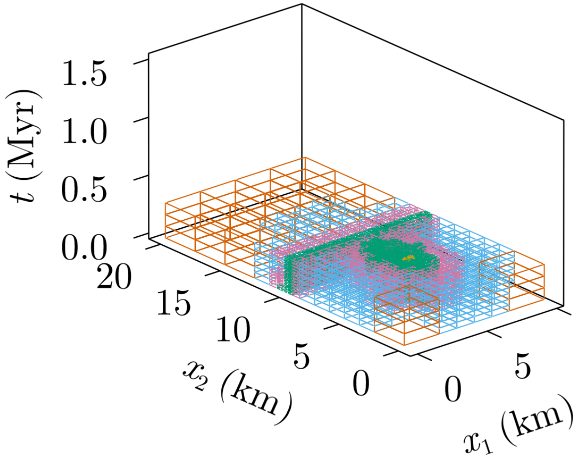

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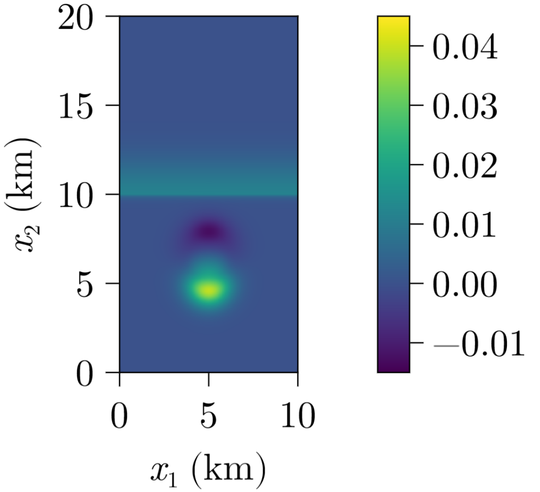

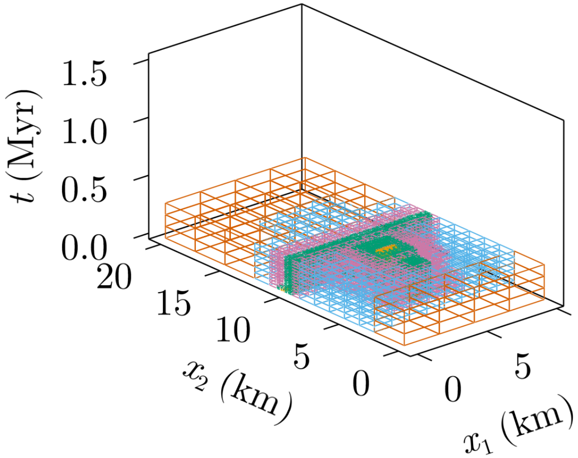

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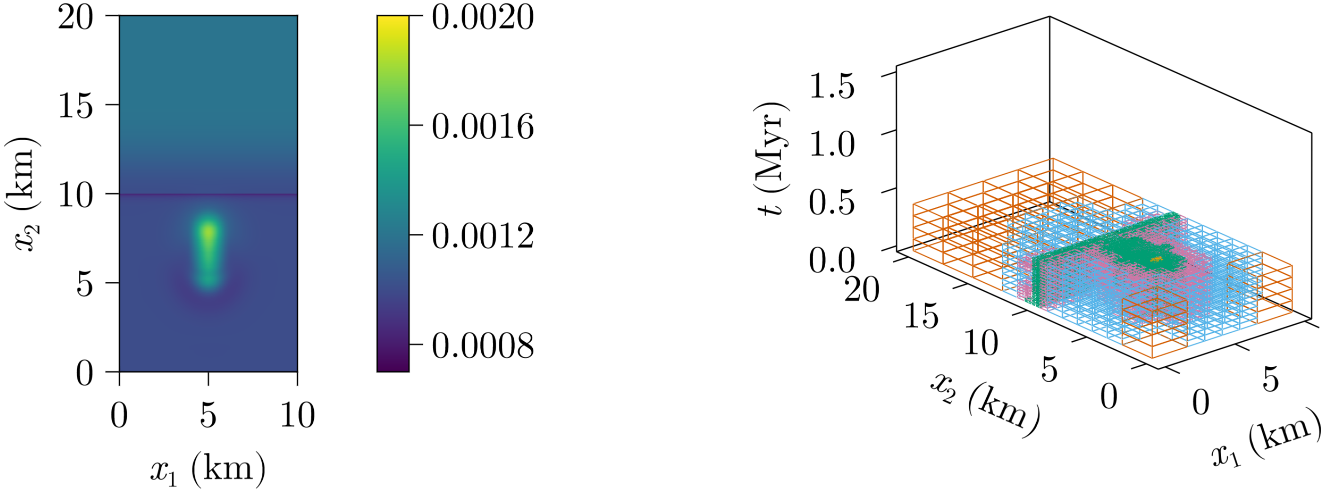

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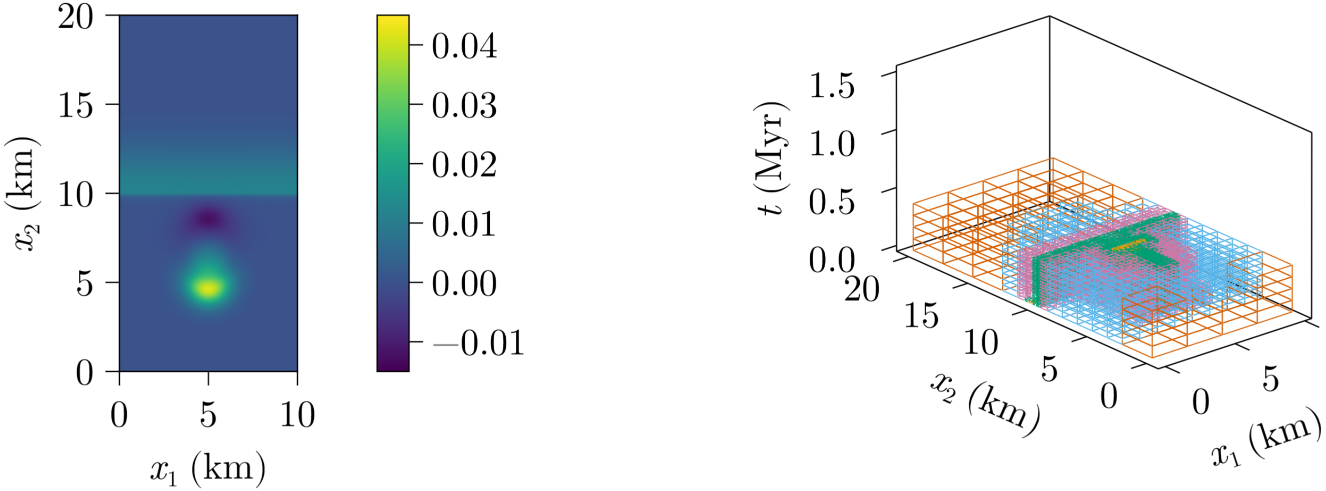

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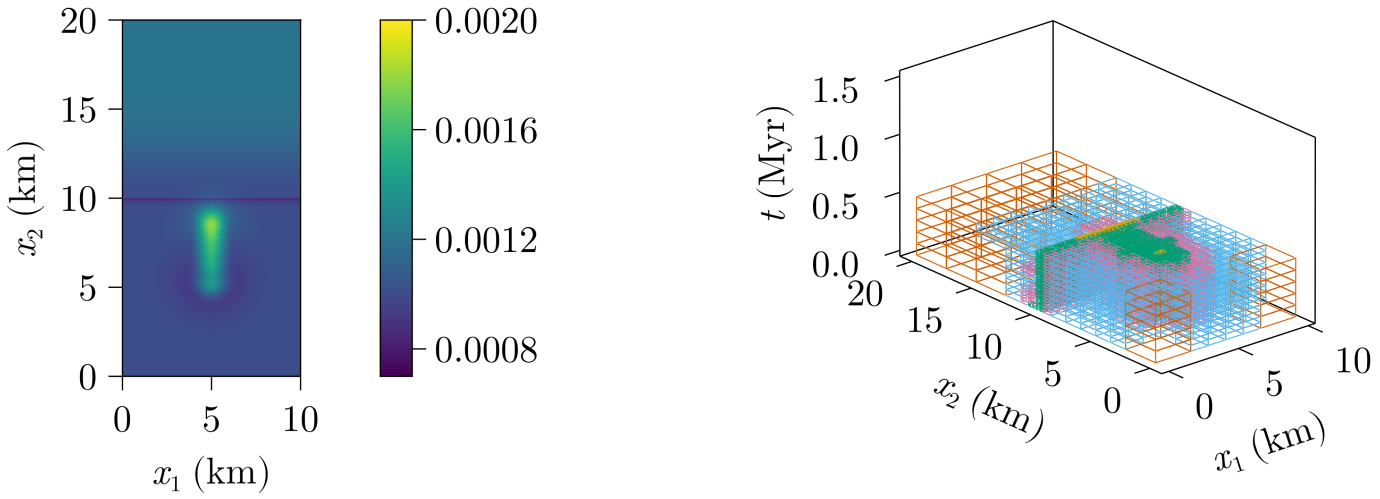

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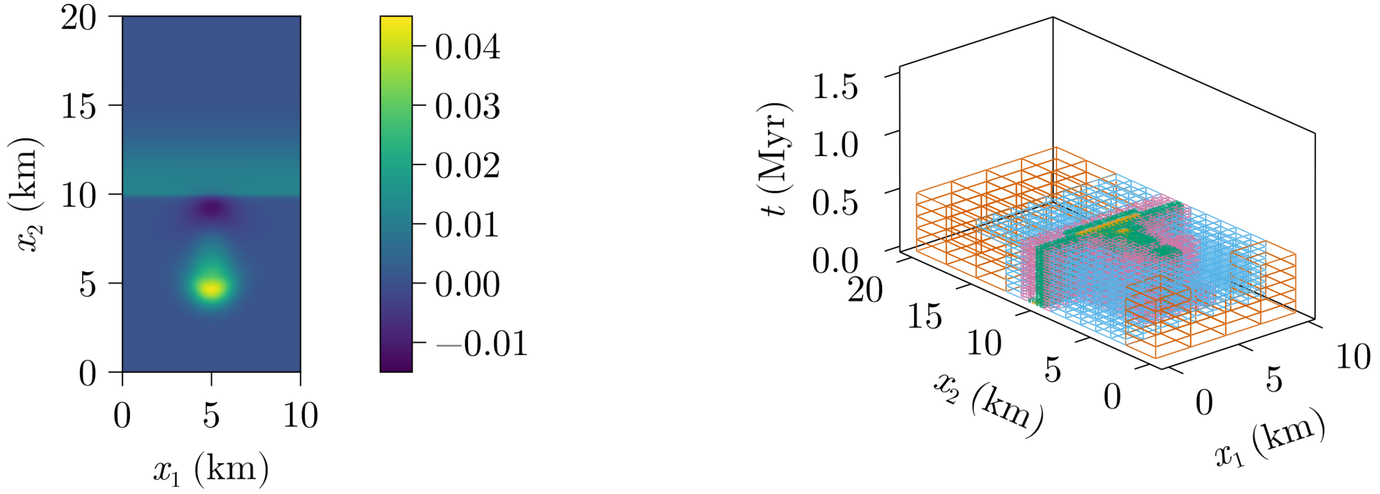

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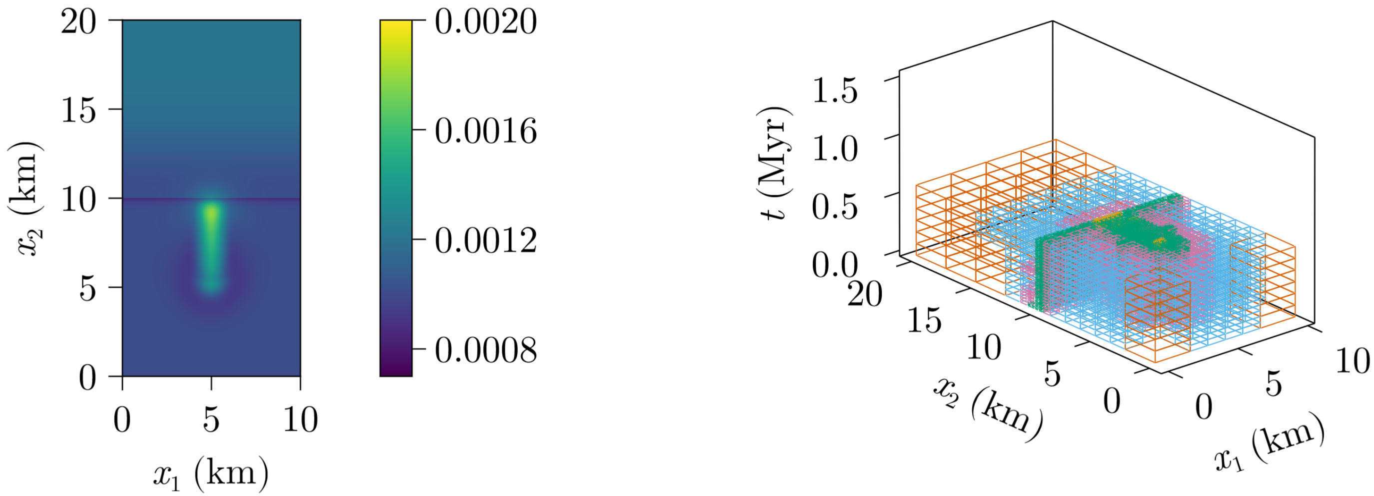

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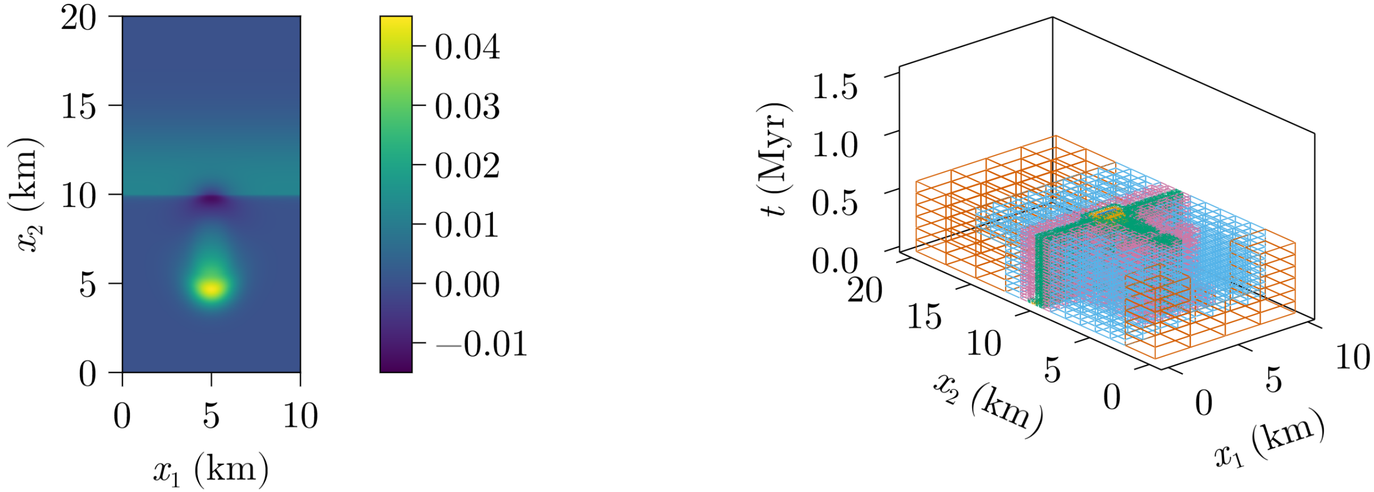

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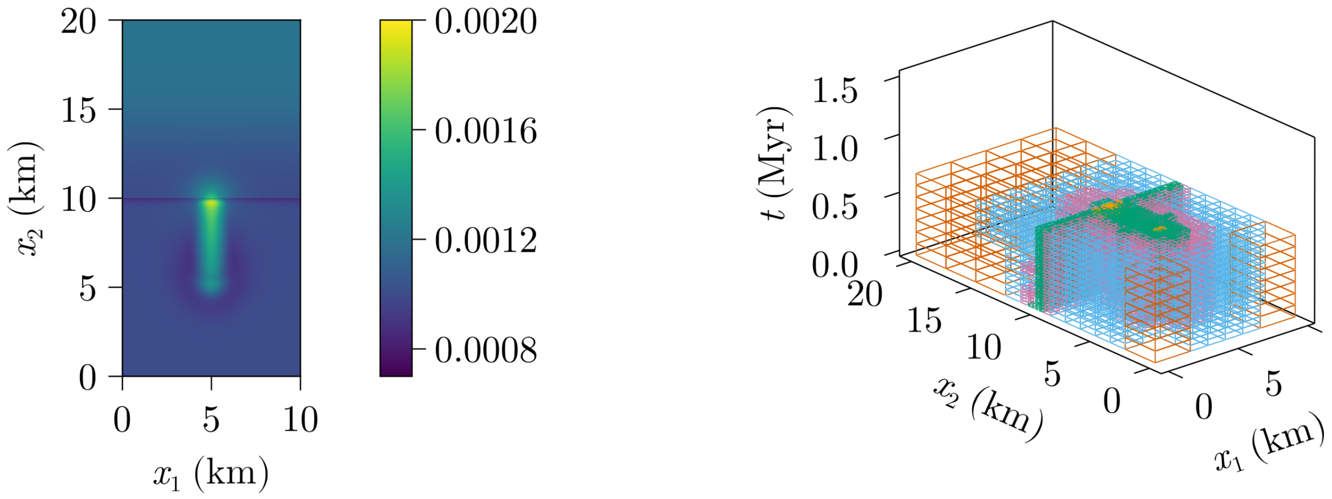

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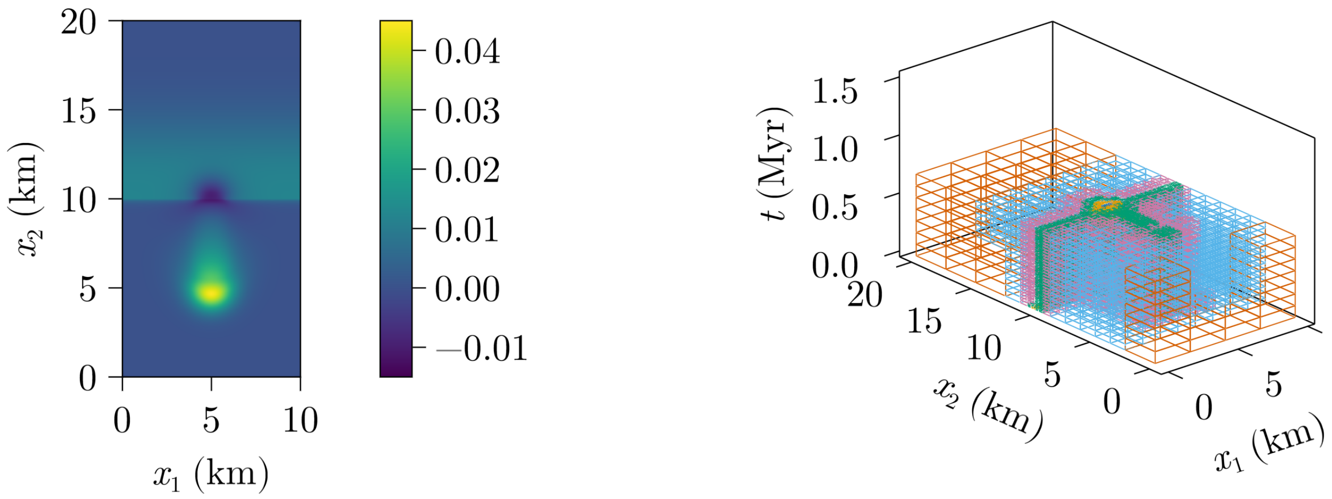

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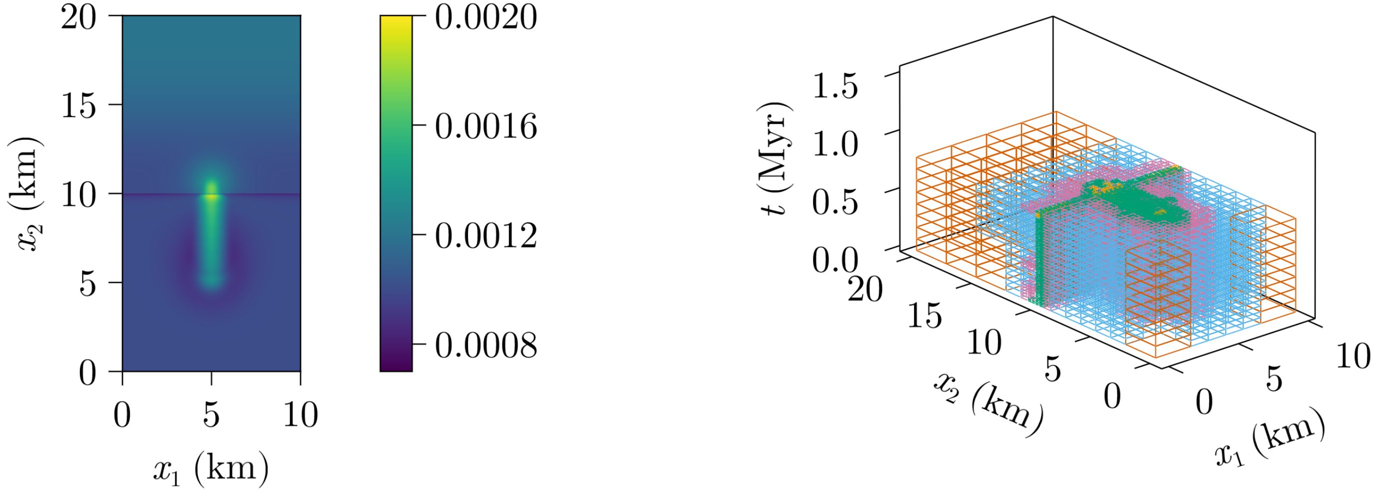

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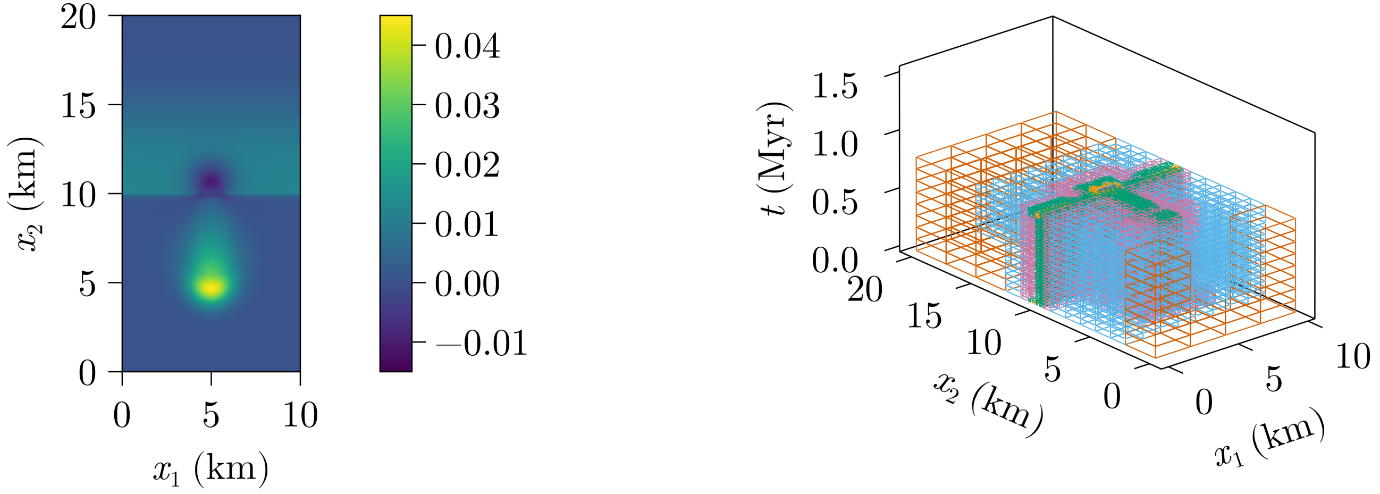

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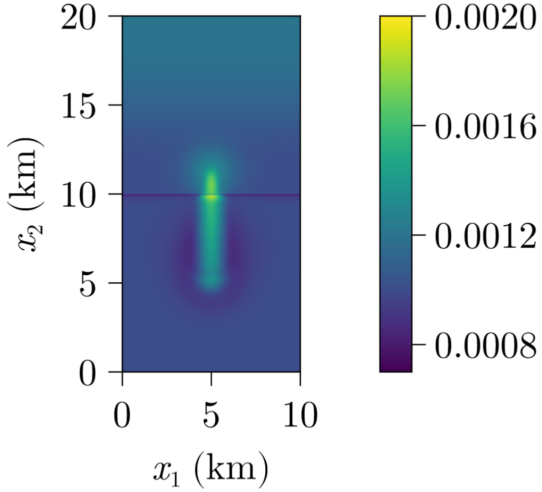

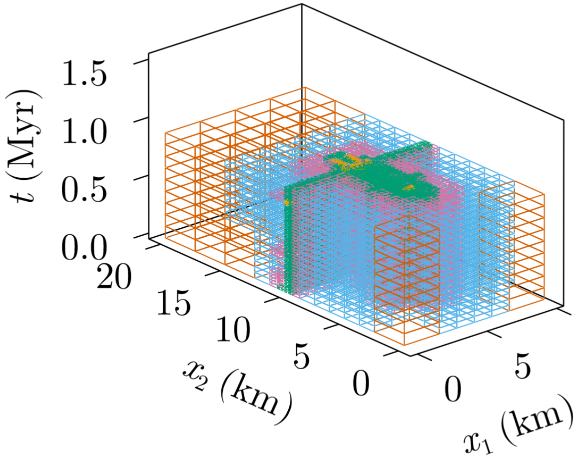

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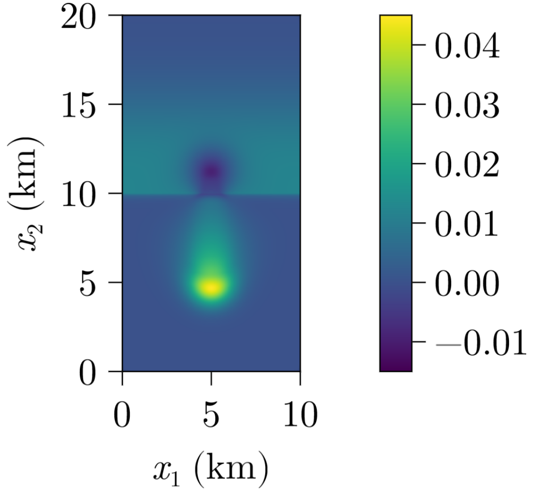

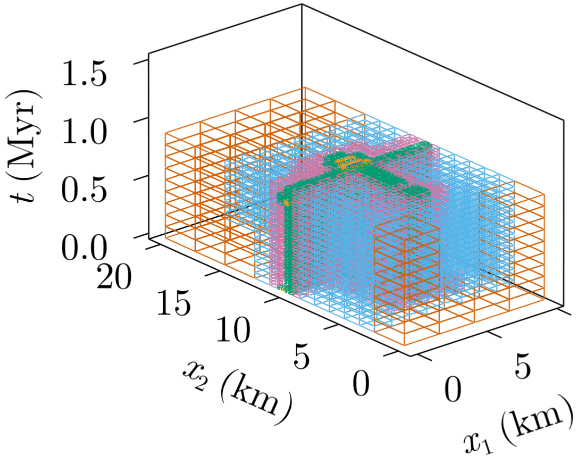

Numerical tests

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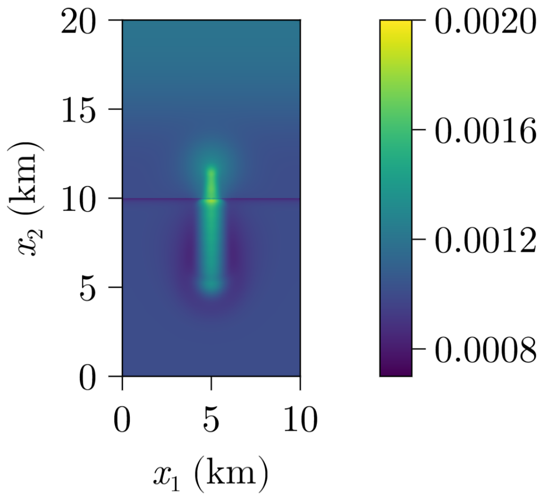

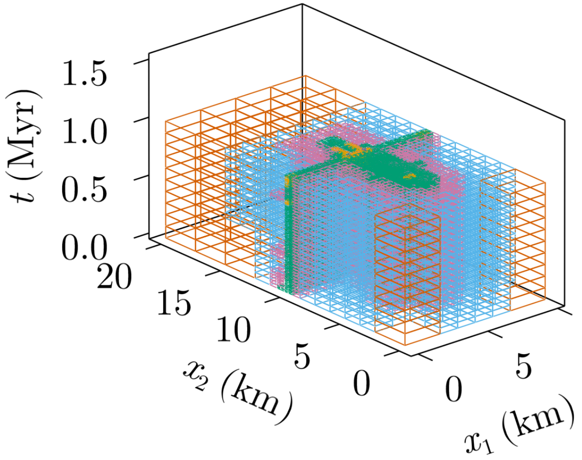

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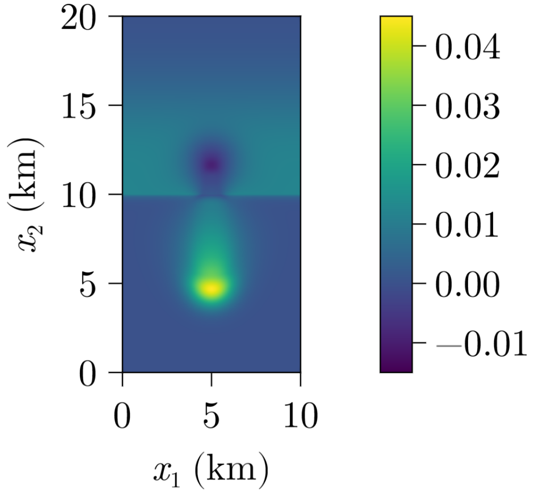

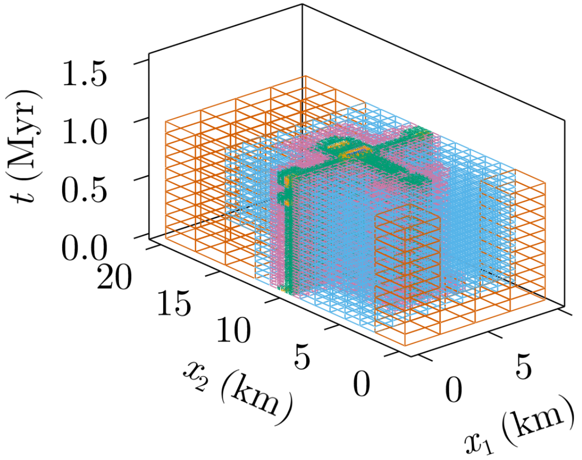

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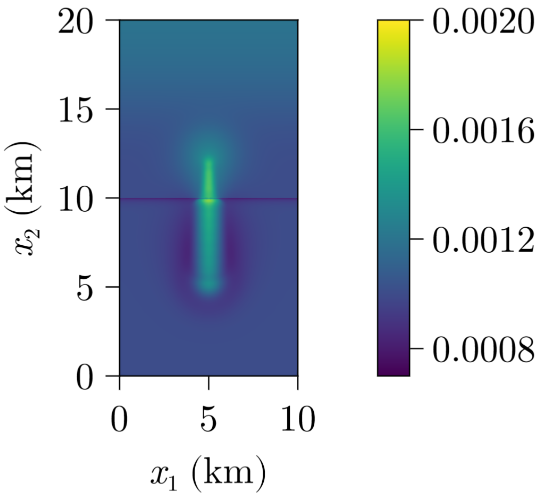

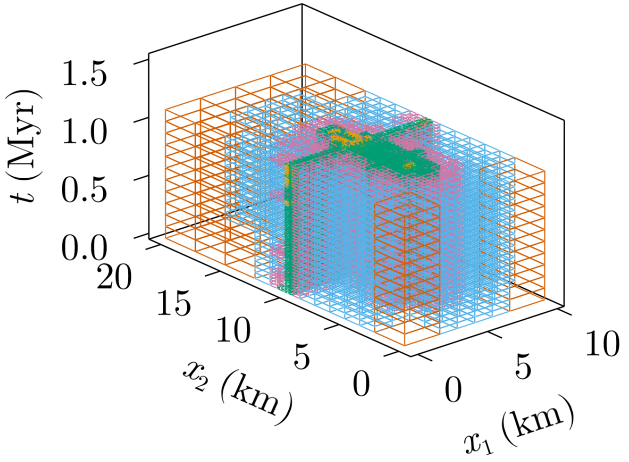

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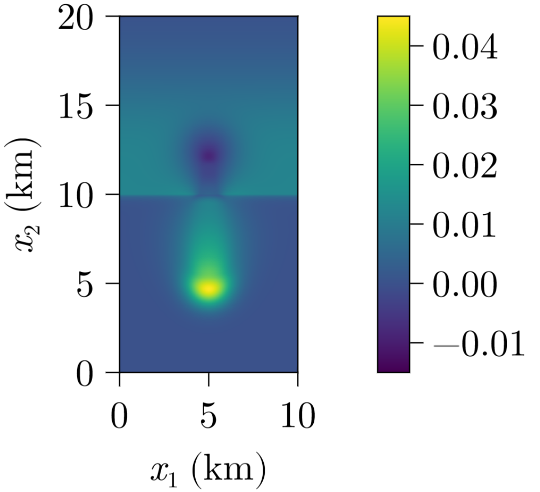

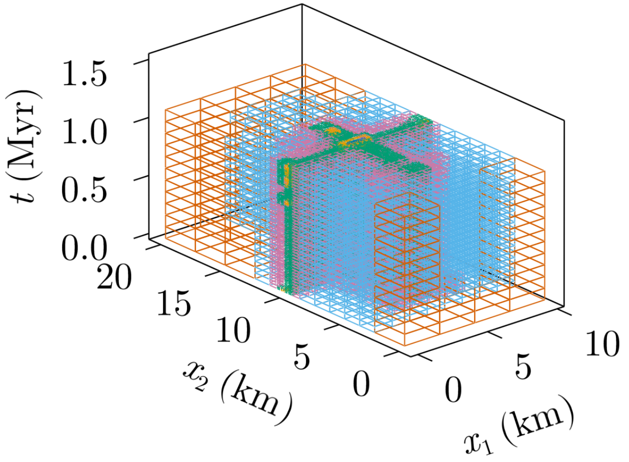

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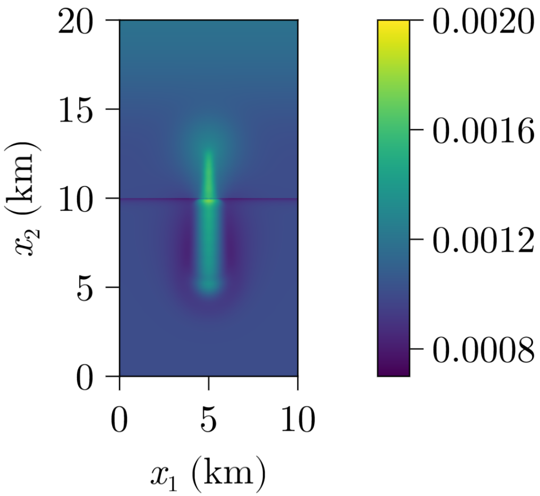

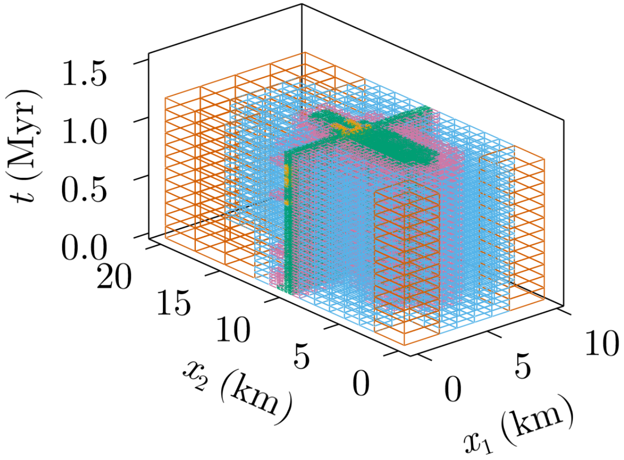

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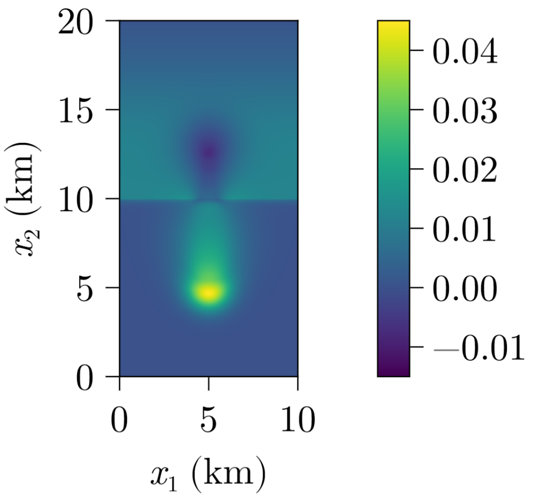

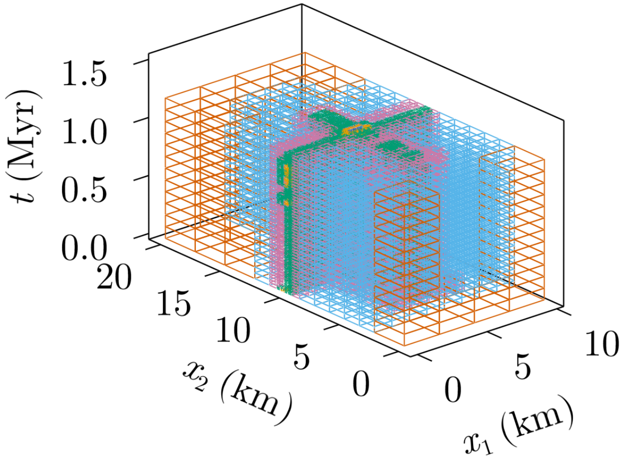

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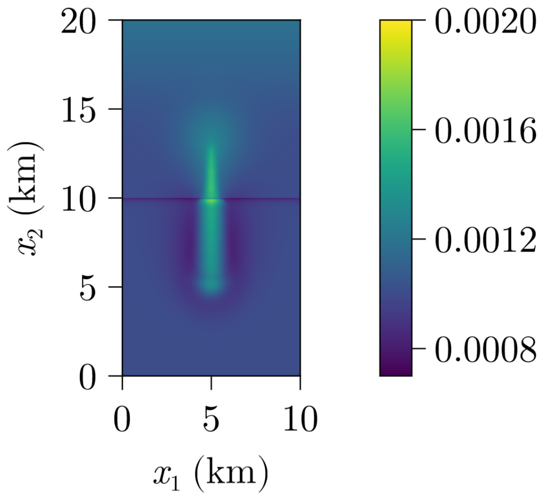

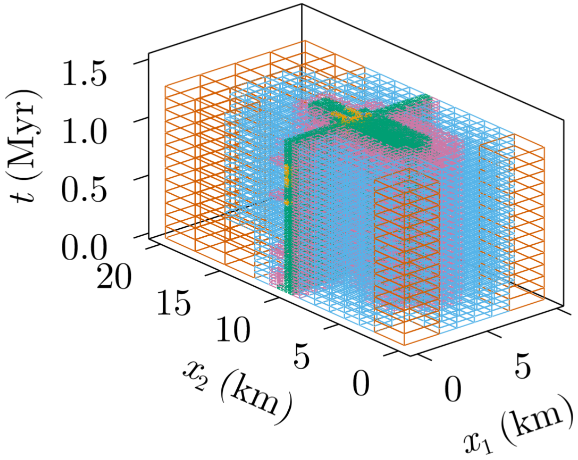

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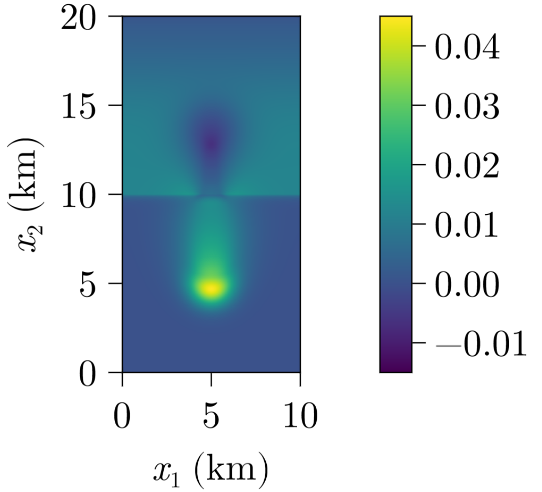

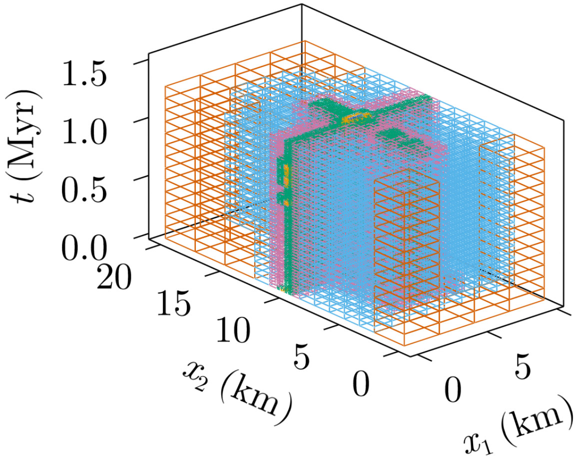

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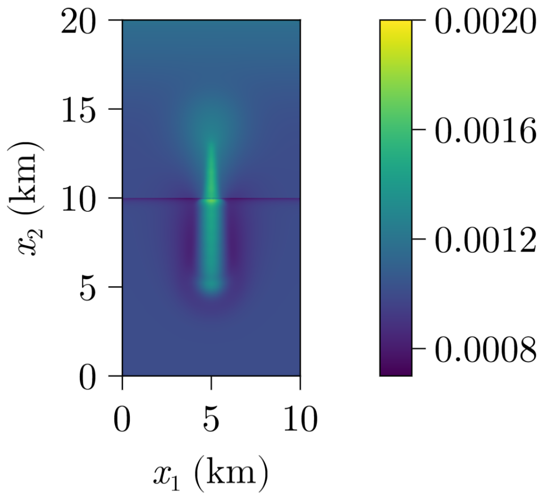

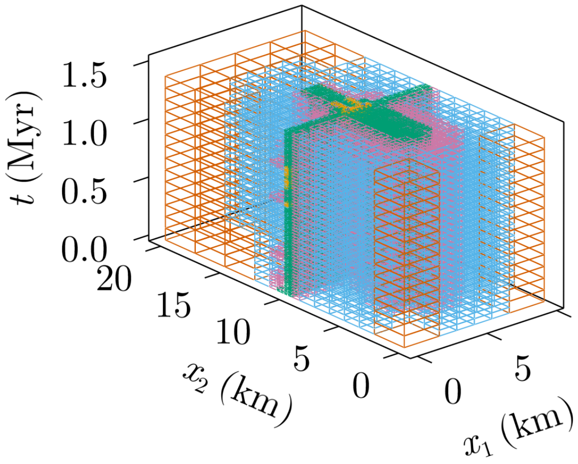

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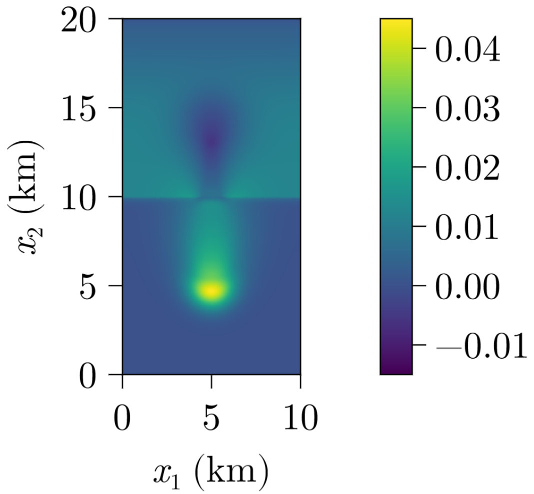

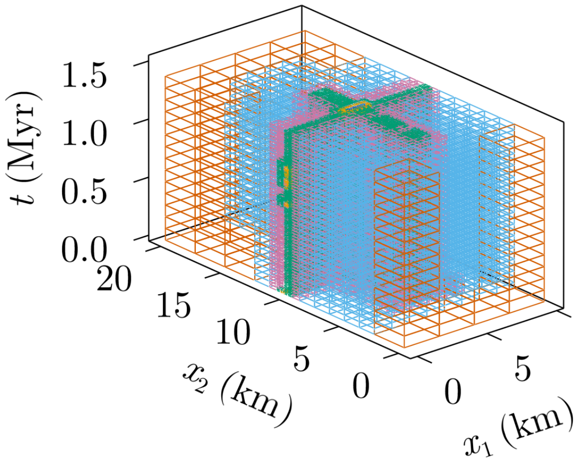

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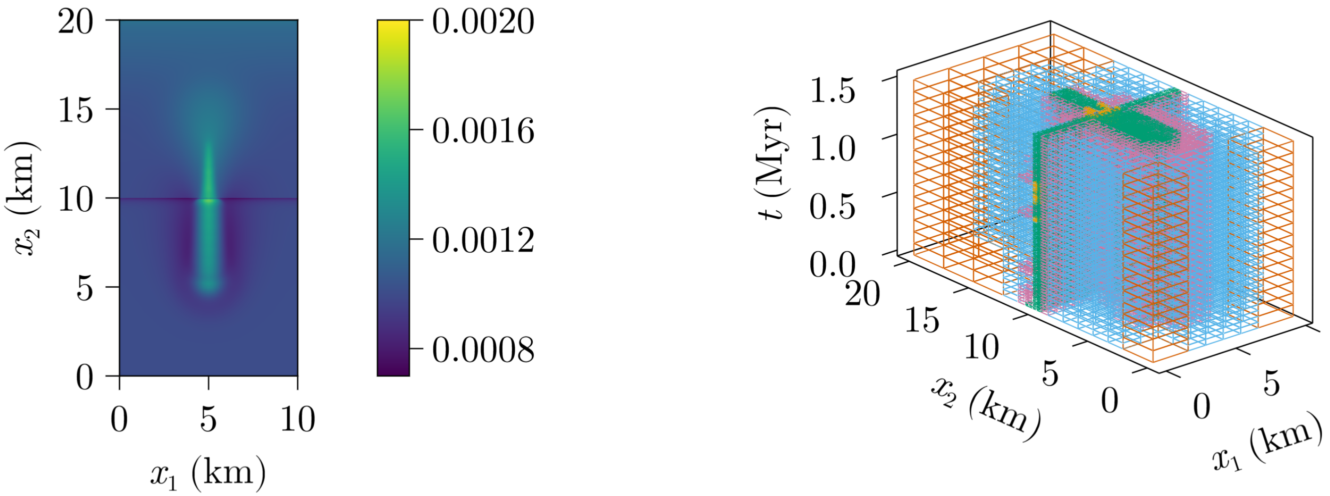

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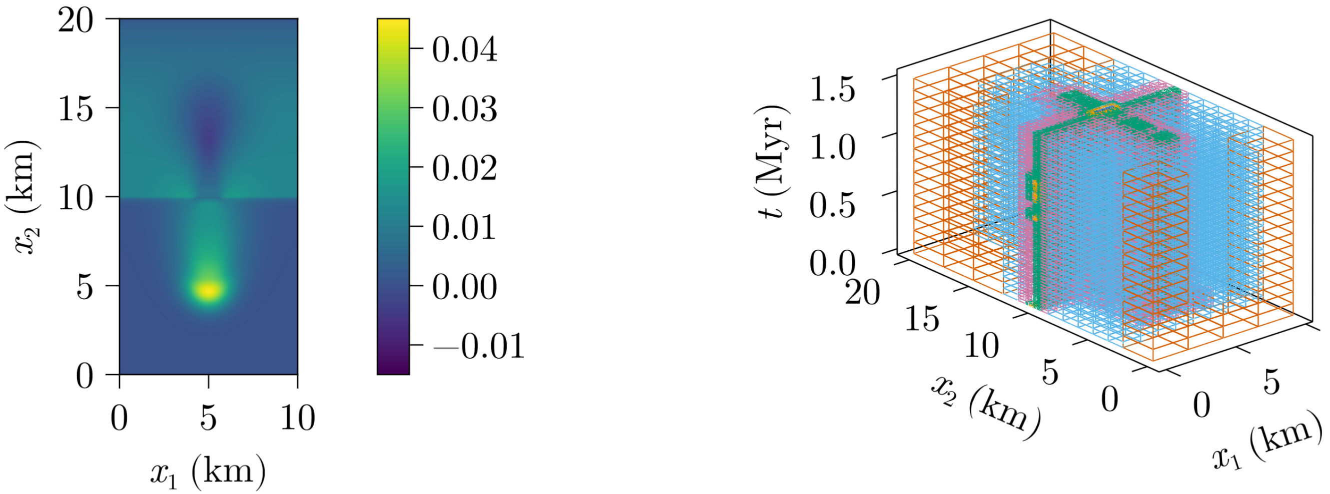

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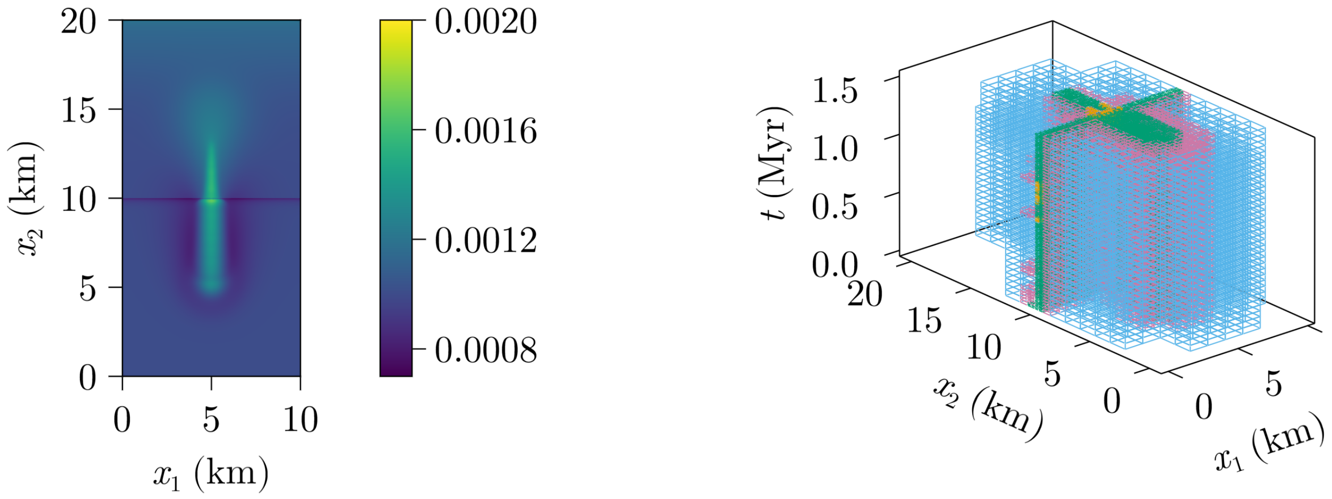

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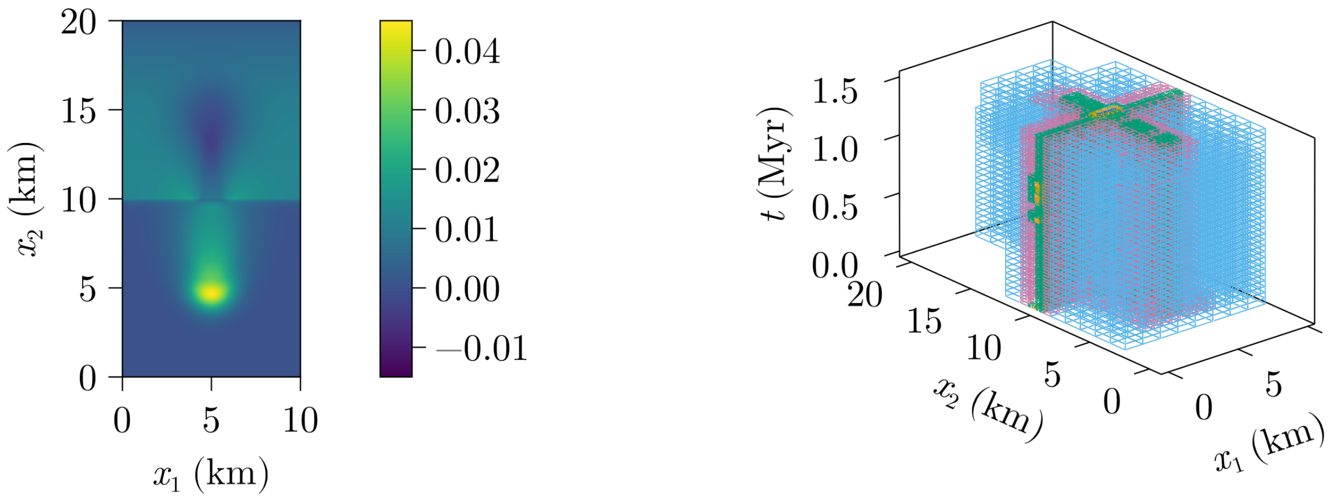

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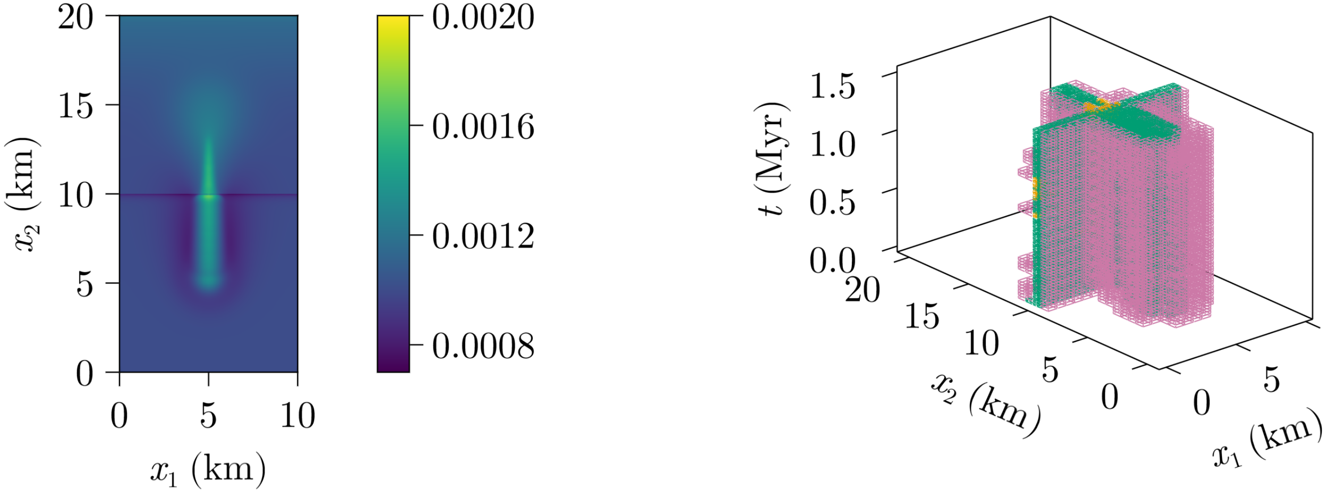

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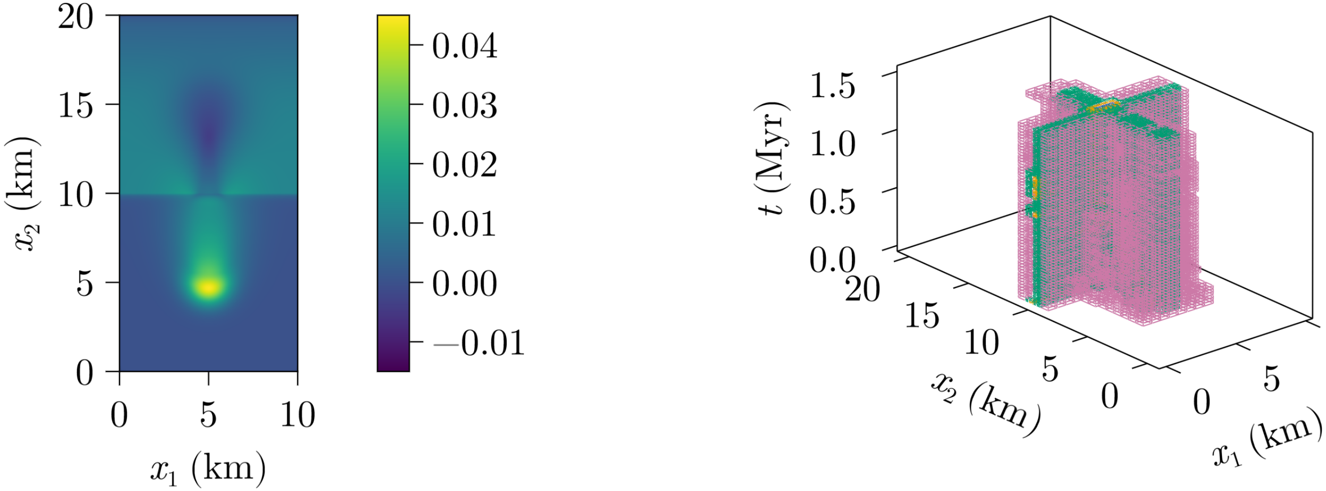

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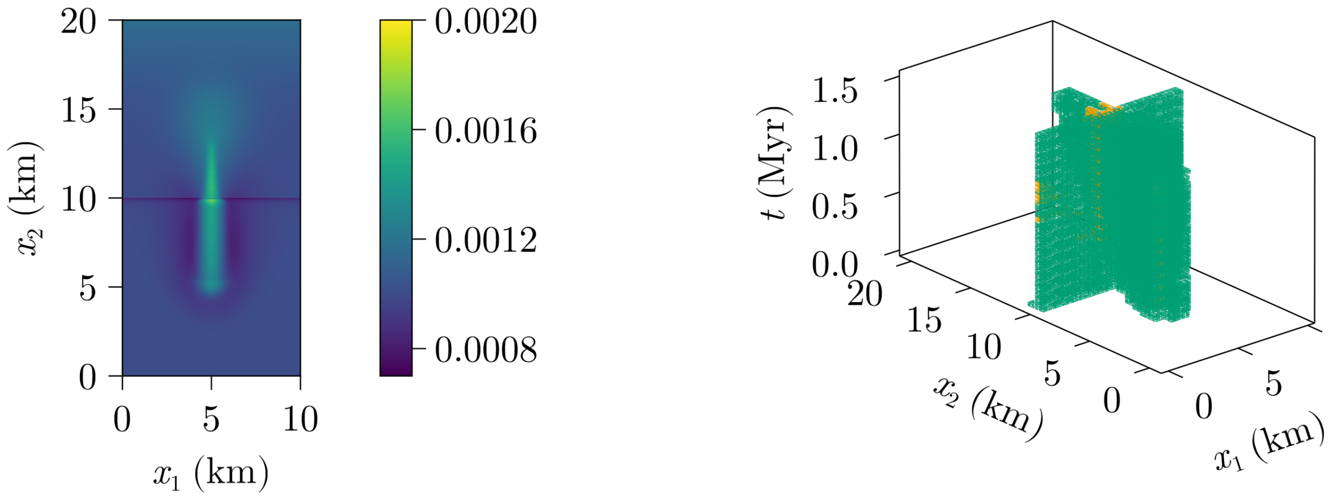

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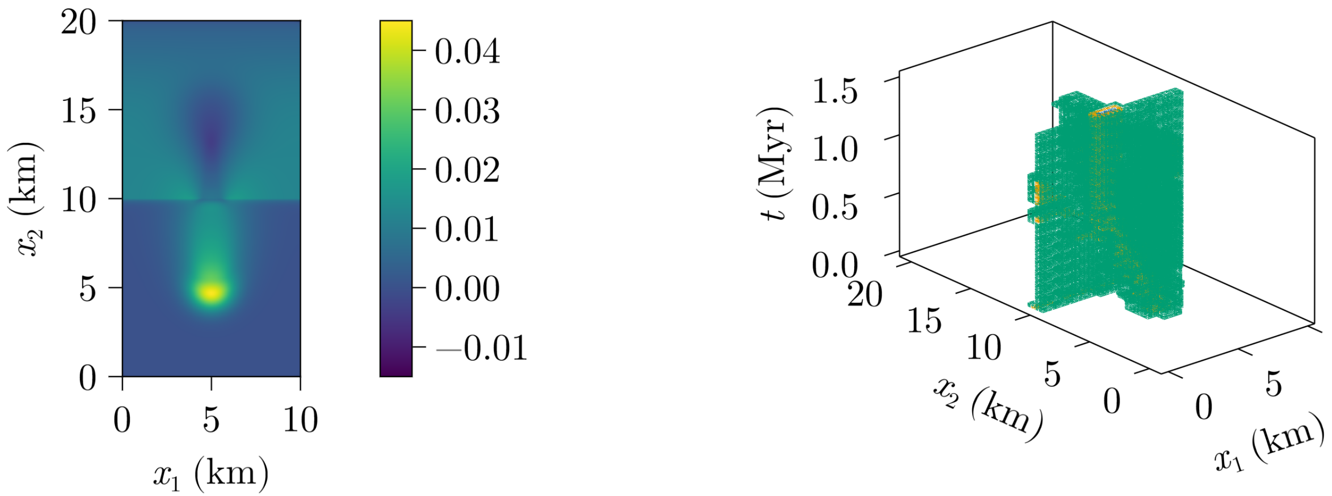

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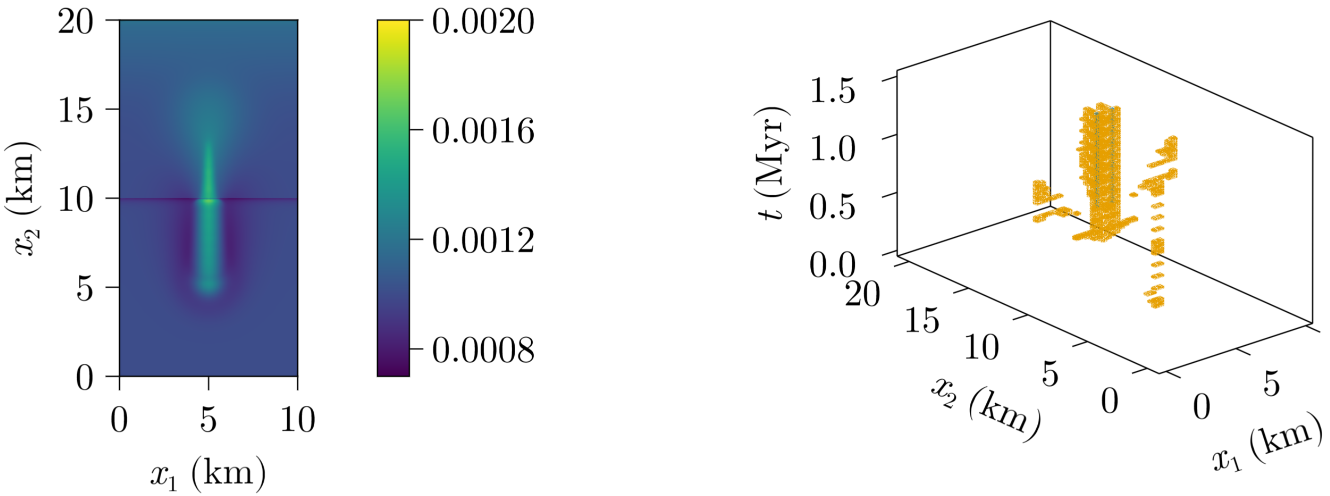

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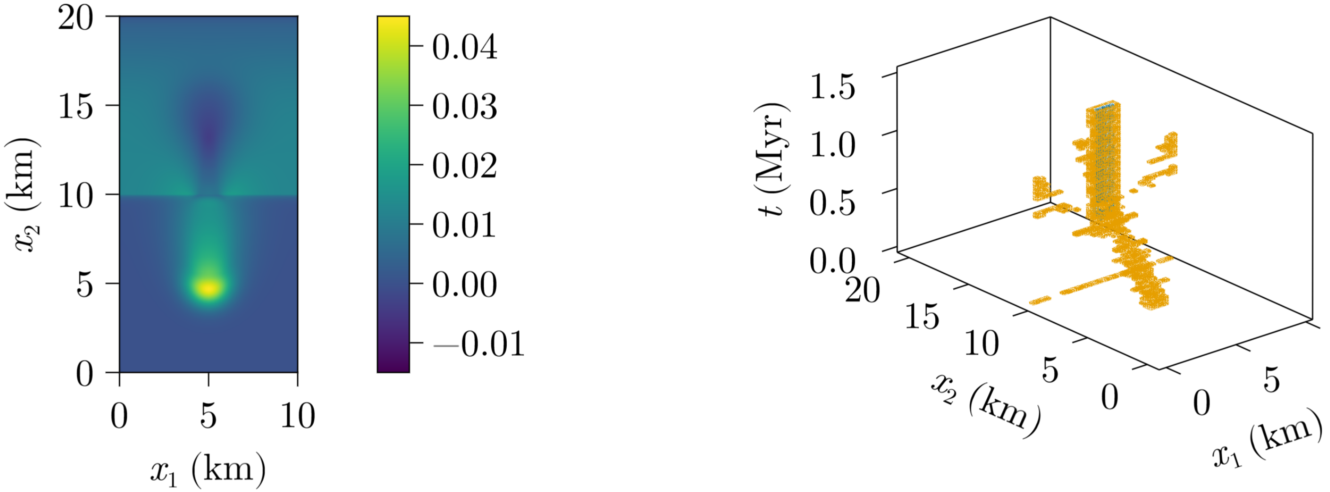

More adaptivity

Perturbed fixed-point iteration argument ⇝ adaptive choice of tolerances:

$$\varphi_\delta^{\text{new}}(t,\cdot) = \Pi\left(\varphi_0 + Q(u_\delta[\varphi_\delta^{\text{old}}](t,\cdot) - u_0) - \int_0^t \mathcal{J}\Big(\beta(\varphi_\delta^{\text{old}}(s,\cdot))\,\kappa(u_\delta[\varphi_\delta^{\text{old}}](s,\cdot))\Big)\,\mathrm{d}s\right).$$

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tol$_u$

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Idea:
- Set tol_φ, guess initial Lipschitz constants L_φ and L_u,
- Choose other tolerances adaptively,
- Optional: Update Lipschitz constants.

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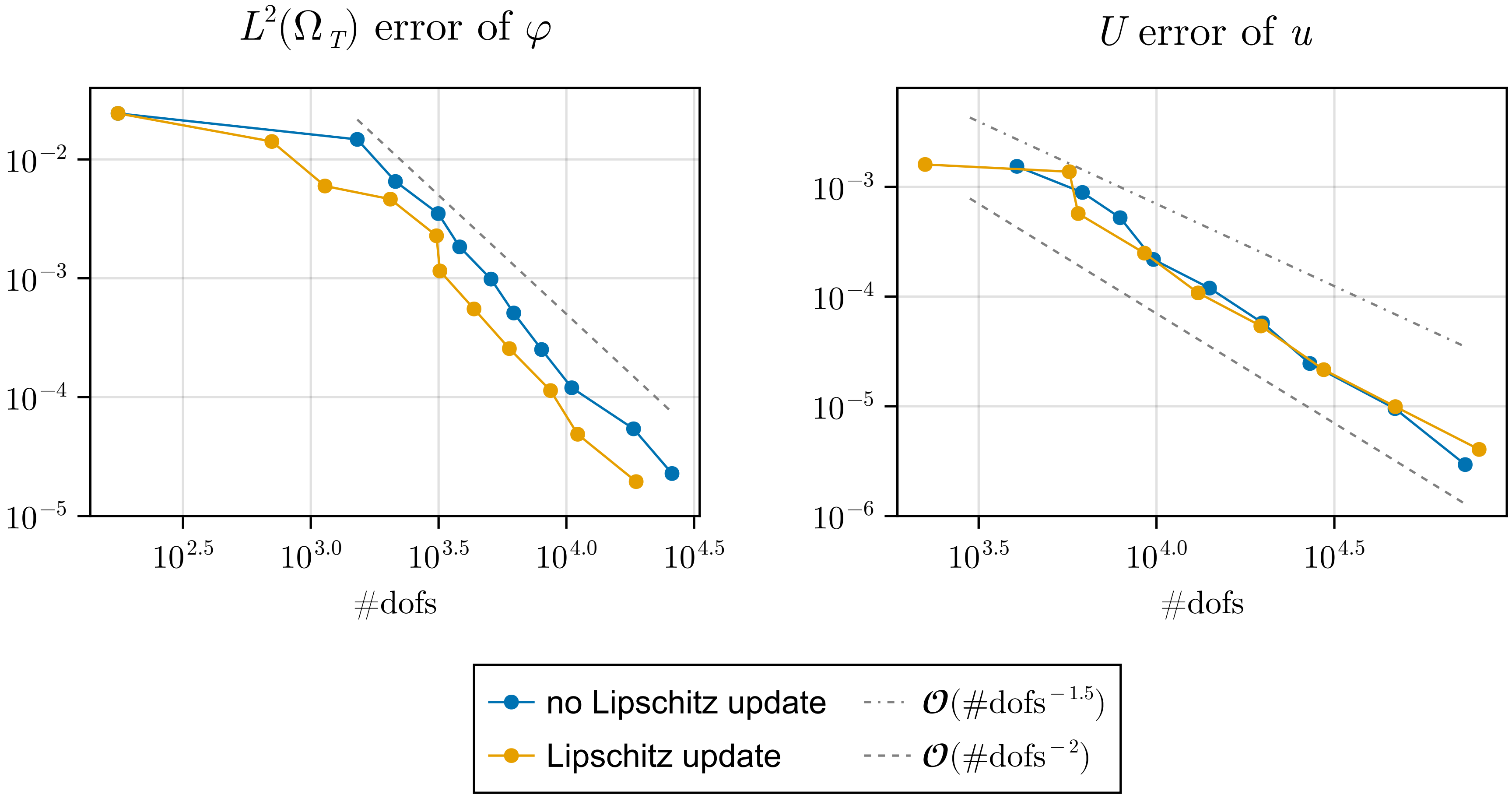
Example: Discontinuous 1d problem from before,
polynomial degree 3 in space and time.

More adaptivity



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Viscous limit

Same method essentially works in the viscous limit case:

- Solve elliptic equation for fixed φ:

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$$G(u, \eta) := \begin{pmatrix} \mathrm{div}_x(\eta) + \tilde{\beta}(\varphi)\frac{u}{\overline{\sigma}} \\ \eta + \tilde{\alpha}(\varphi)\, \nabla_x u \end{pmatrix} = \begin{pmatrix} 0 \\ -\tilde{\alpha}(\varphi)\zeta(\varphi) \end{pmatrix} =: R ,$$

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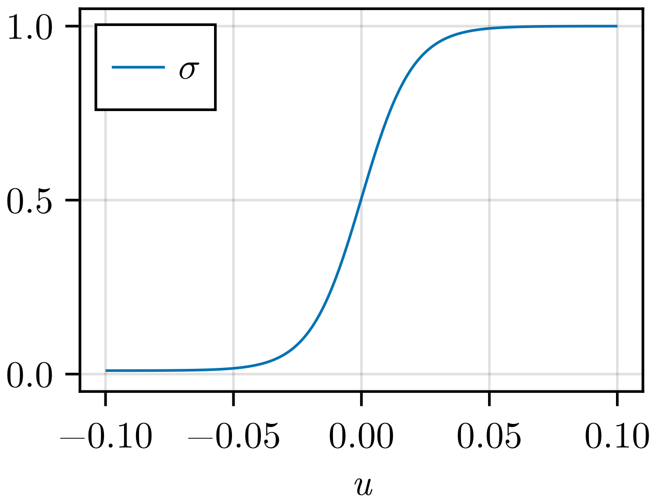
Viscous limit

“Realistic” parameter choice:

$$\Omega = (0\,,20)\ \text{km}$$
$$T = 1.5\ \text{Myr}$$
$$\alpha(\varphi) = 1000\,(1 - \exp(-\varphi))^3$$
$$\beta(\varphi) = 1 - \exp(-\varphi)$$
$$\zeta(\varphi) = \exp(-\varphi)$$
$$\sigma(u) = \frac{10^{-2} + \exp(10^2\,u)}{1 + \exp(10^2\,u)}$$
$$Q = 0$$



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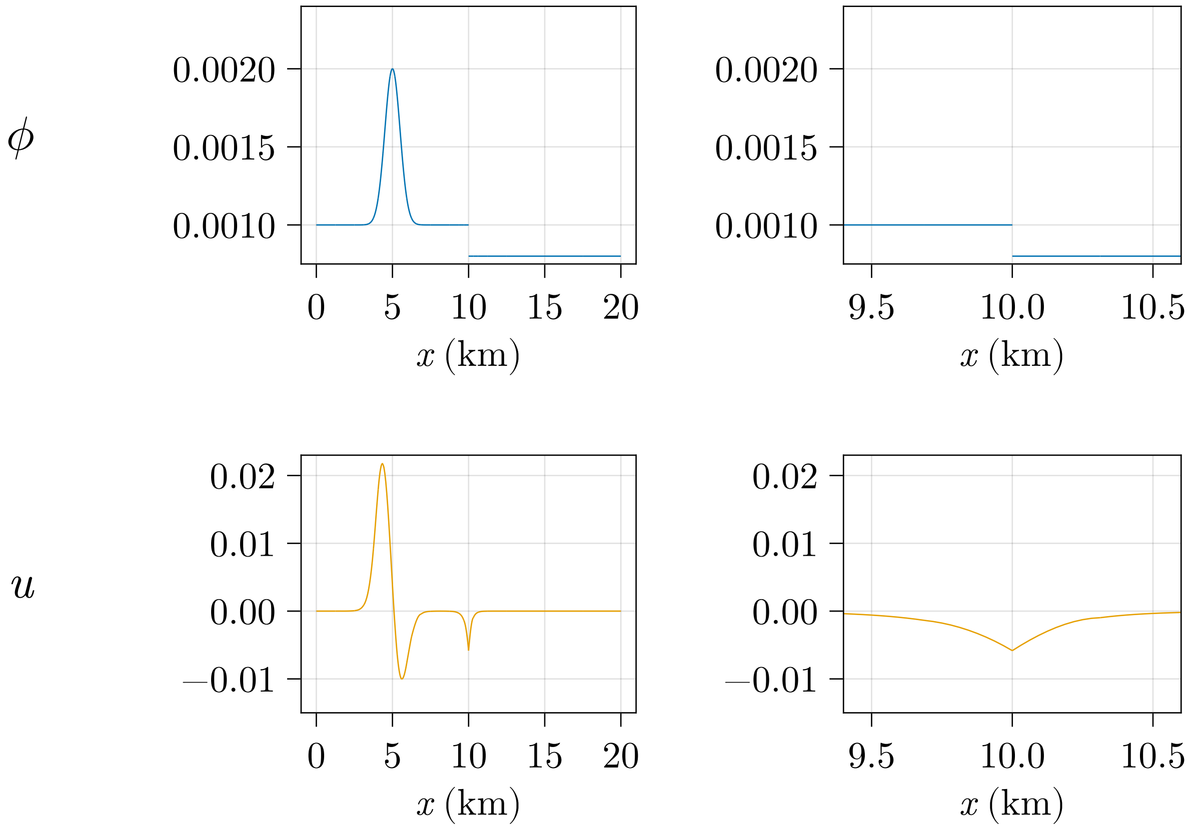

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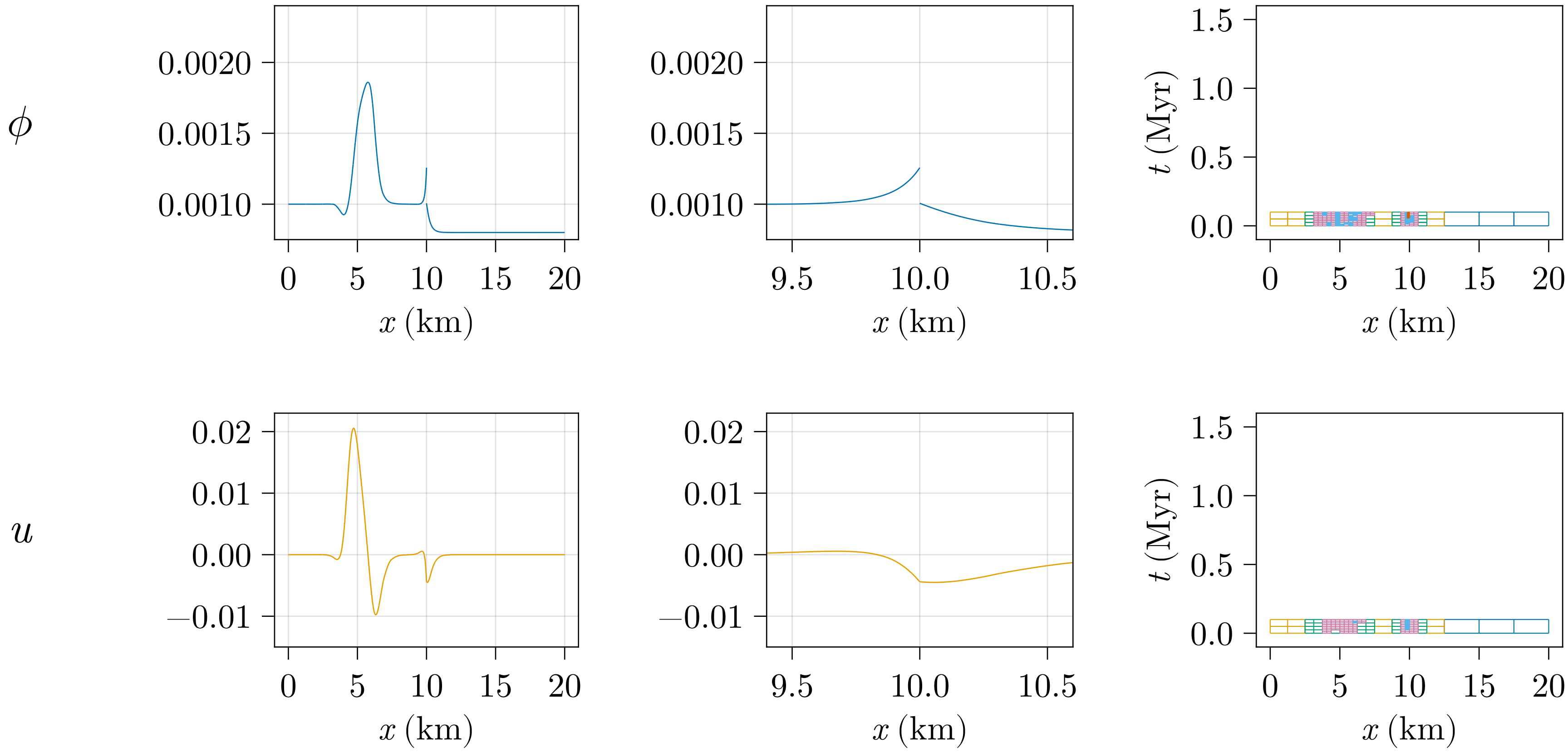

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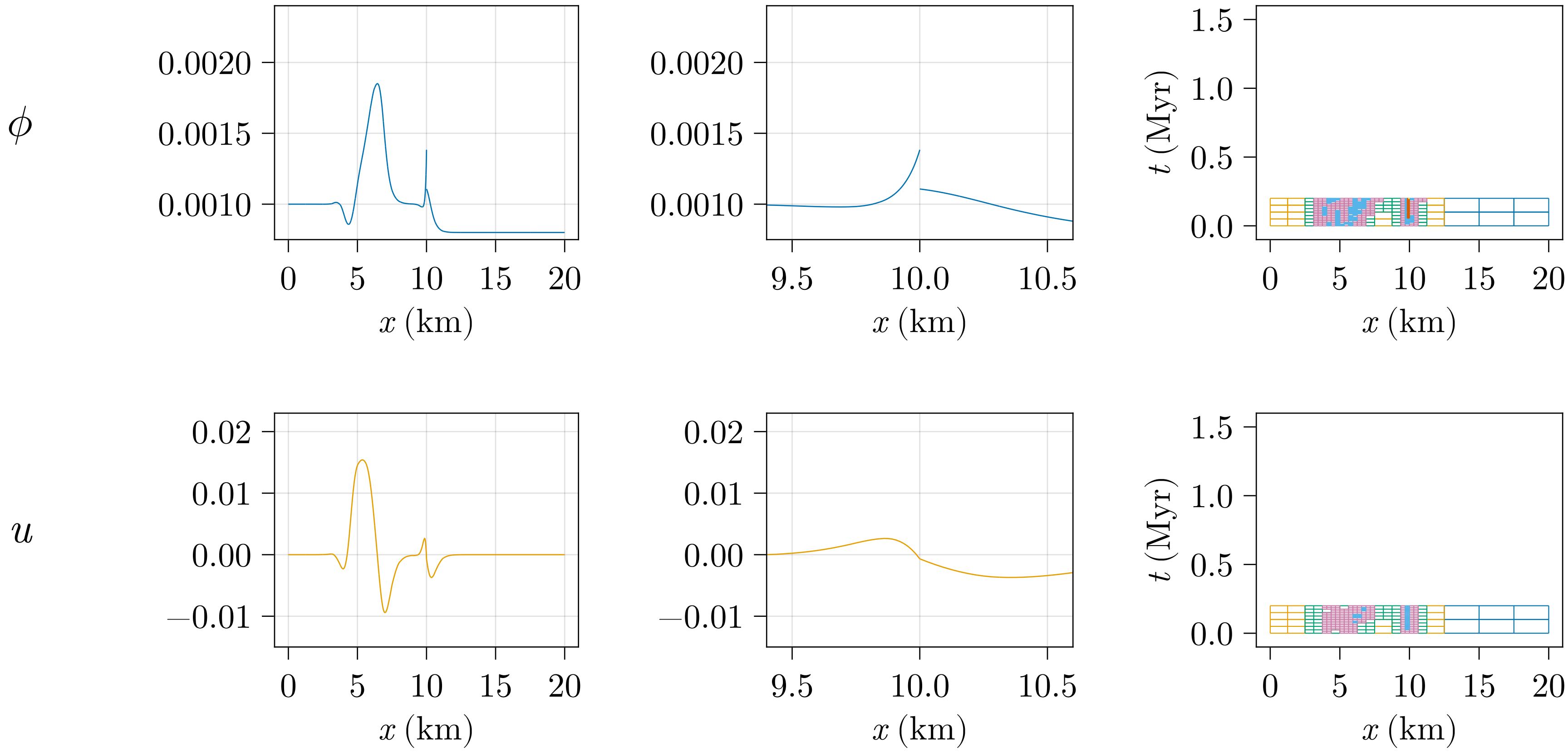

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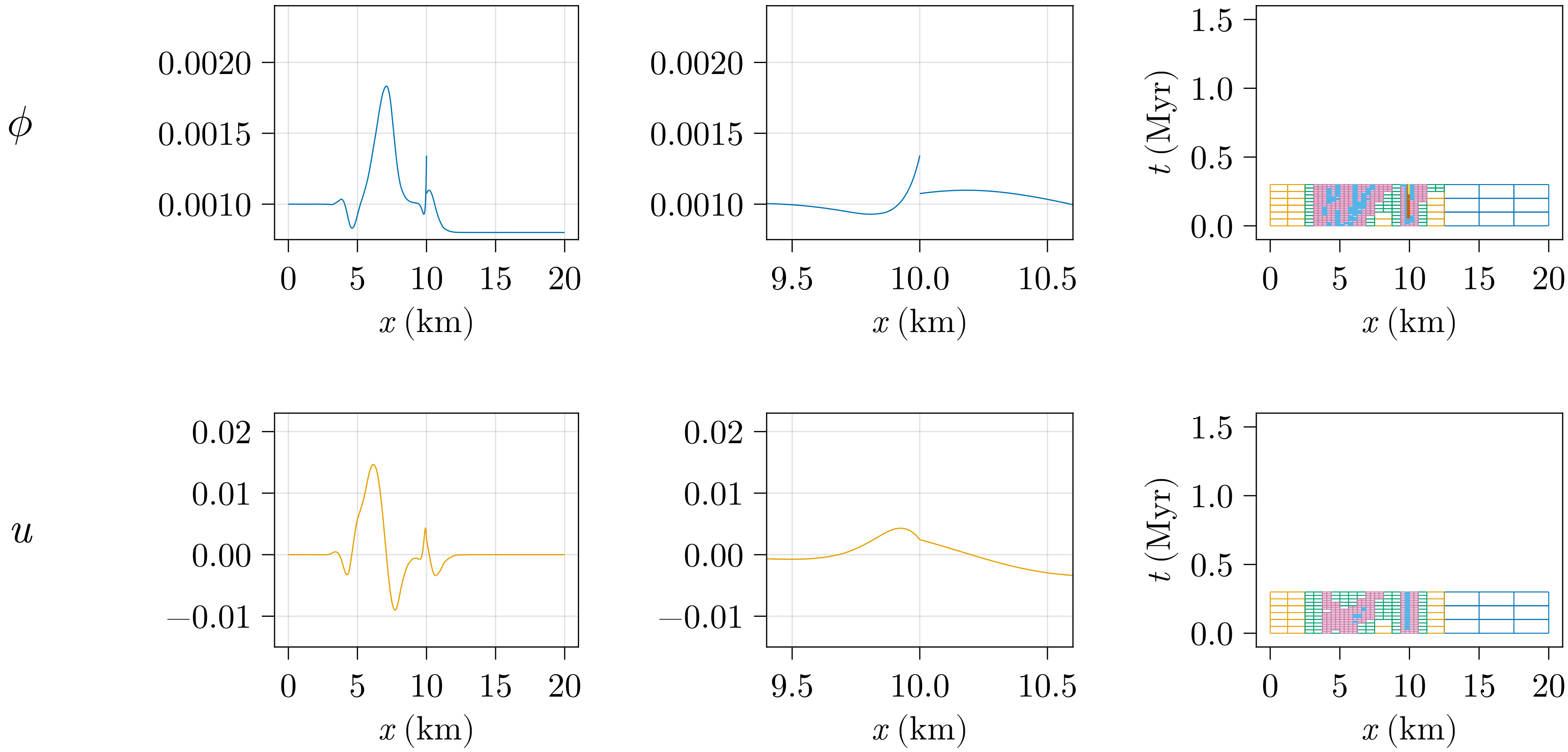

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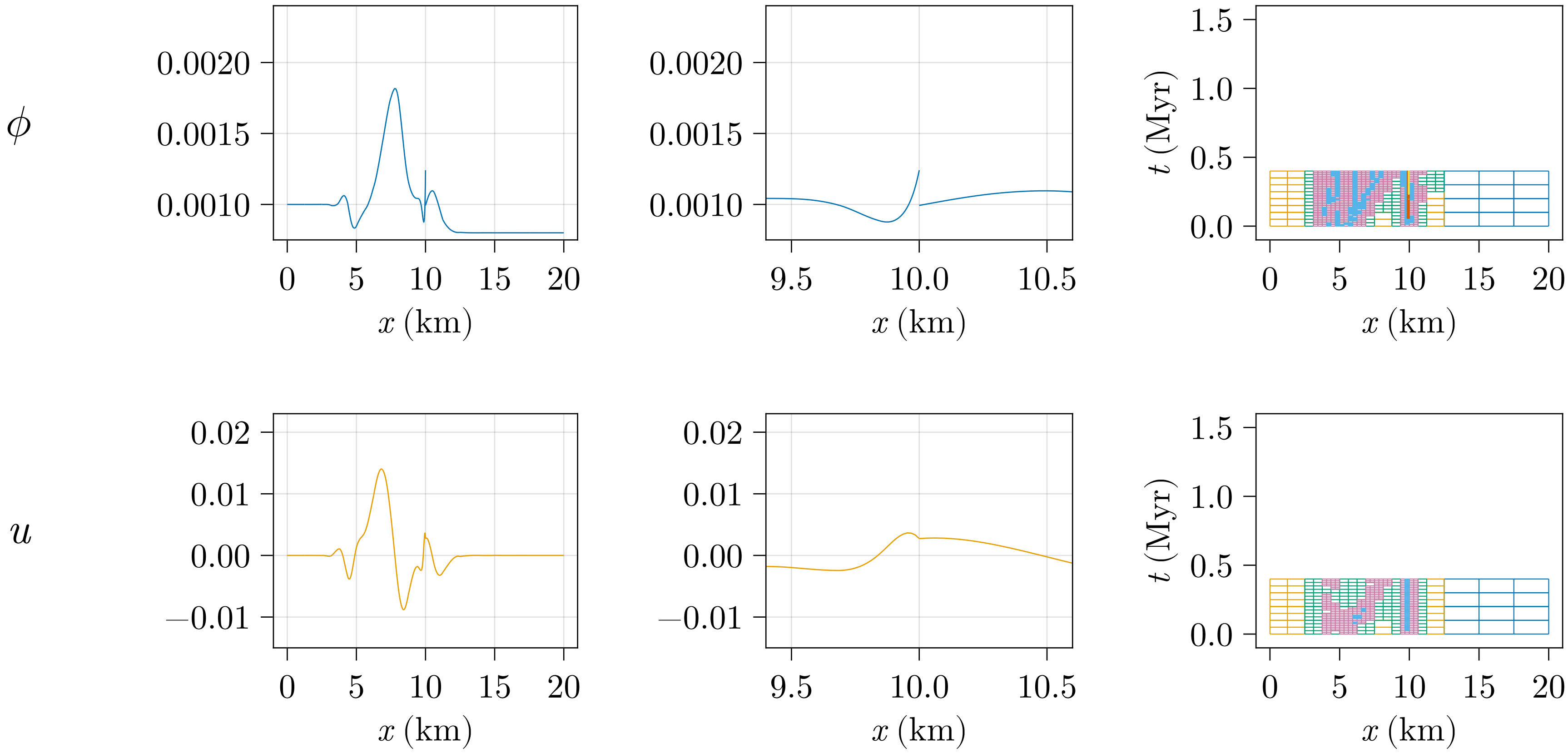

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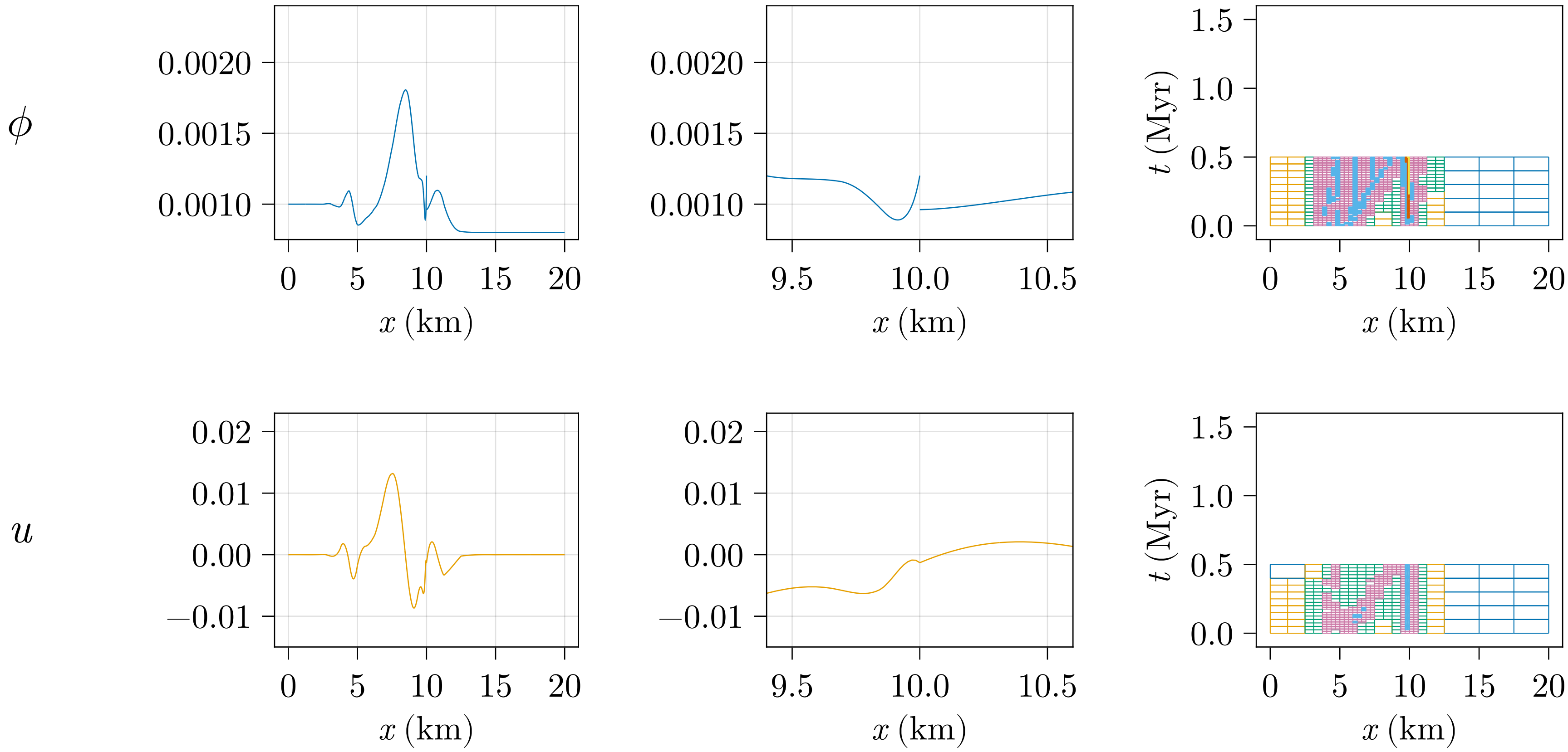
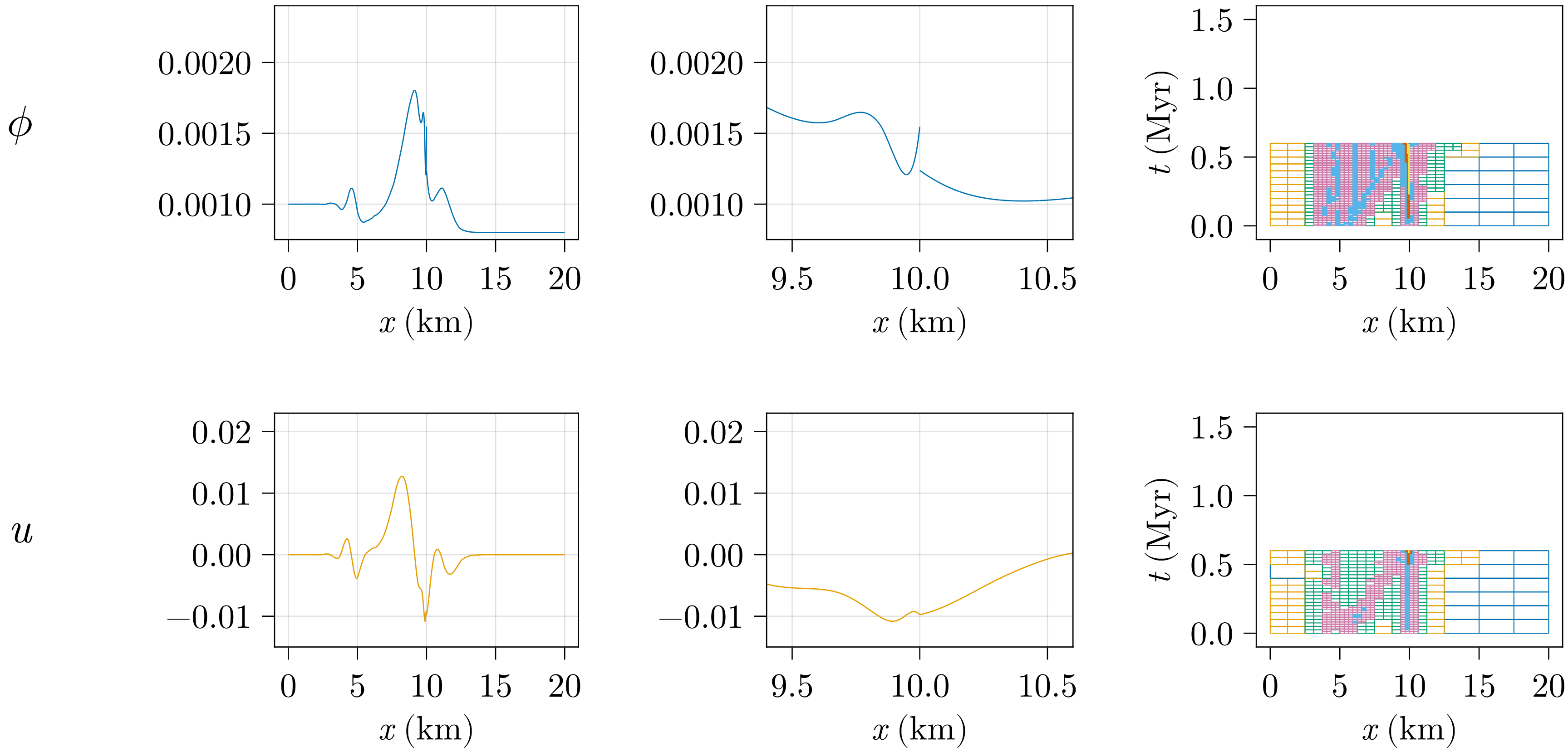


Viscous limit



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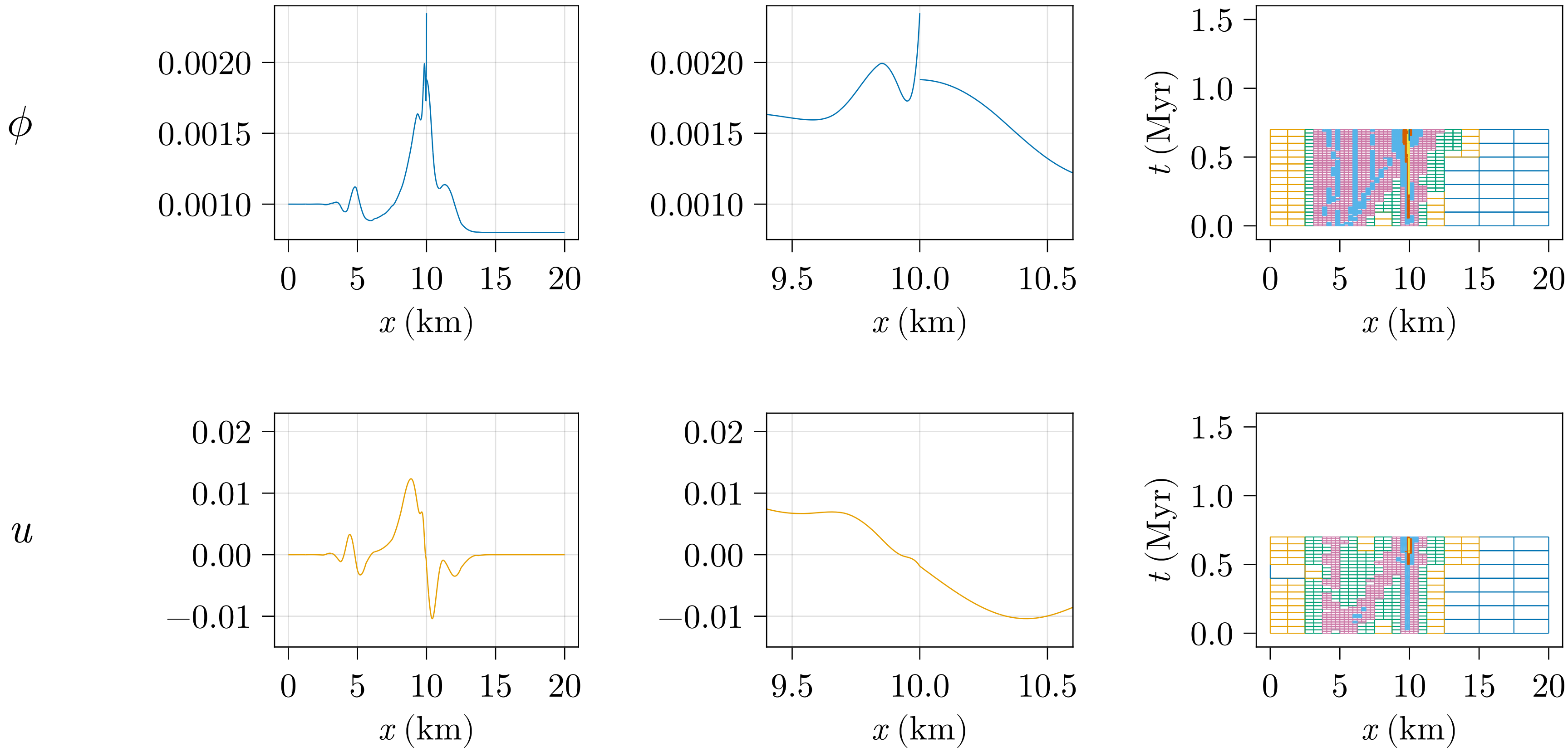

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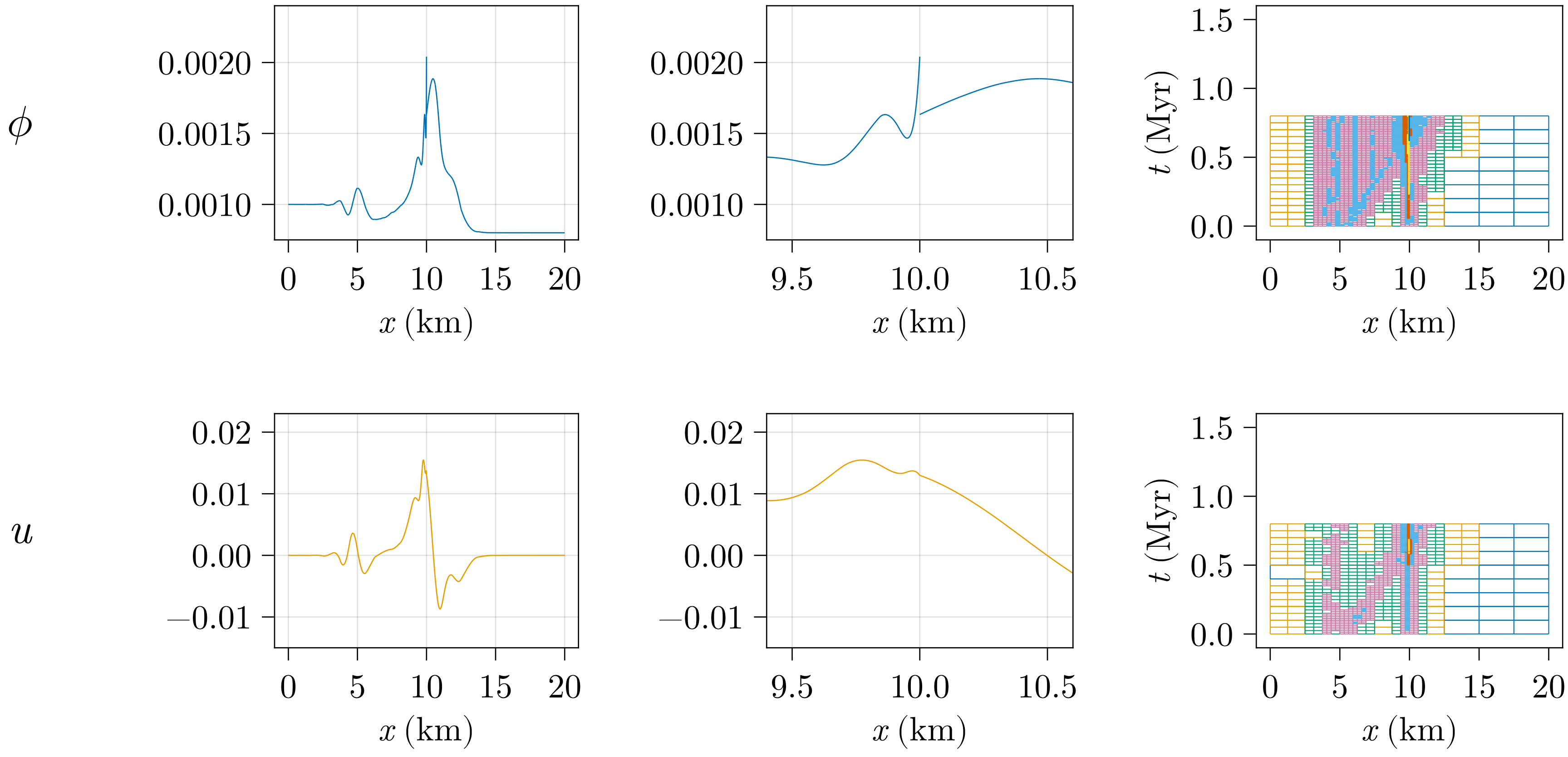

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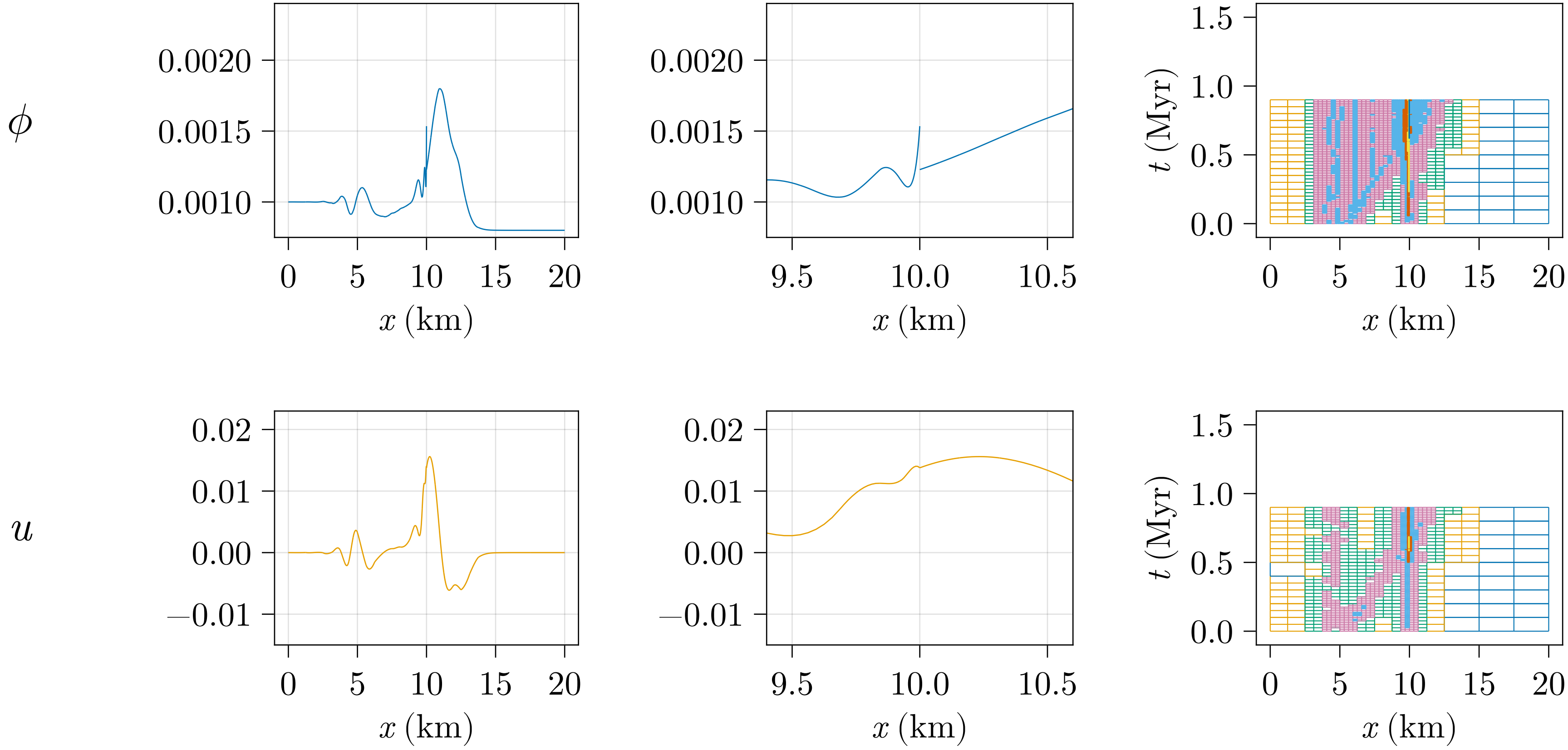

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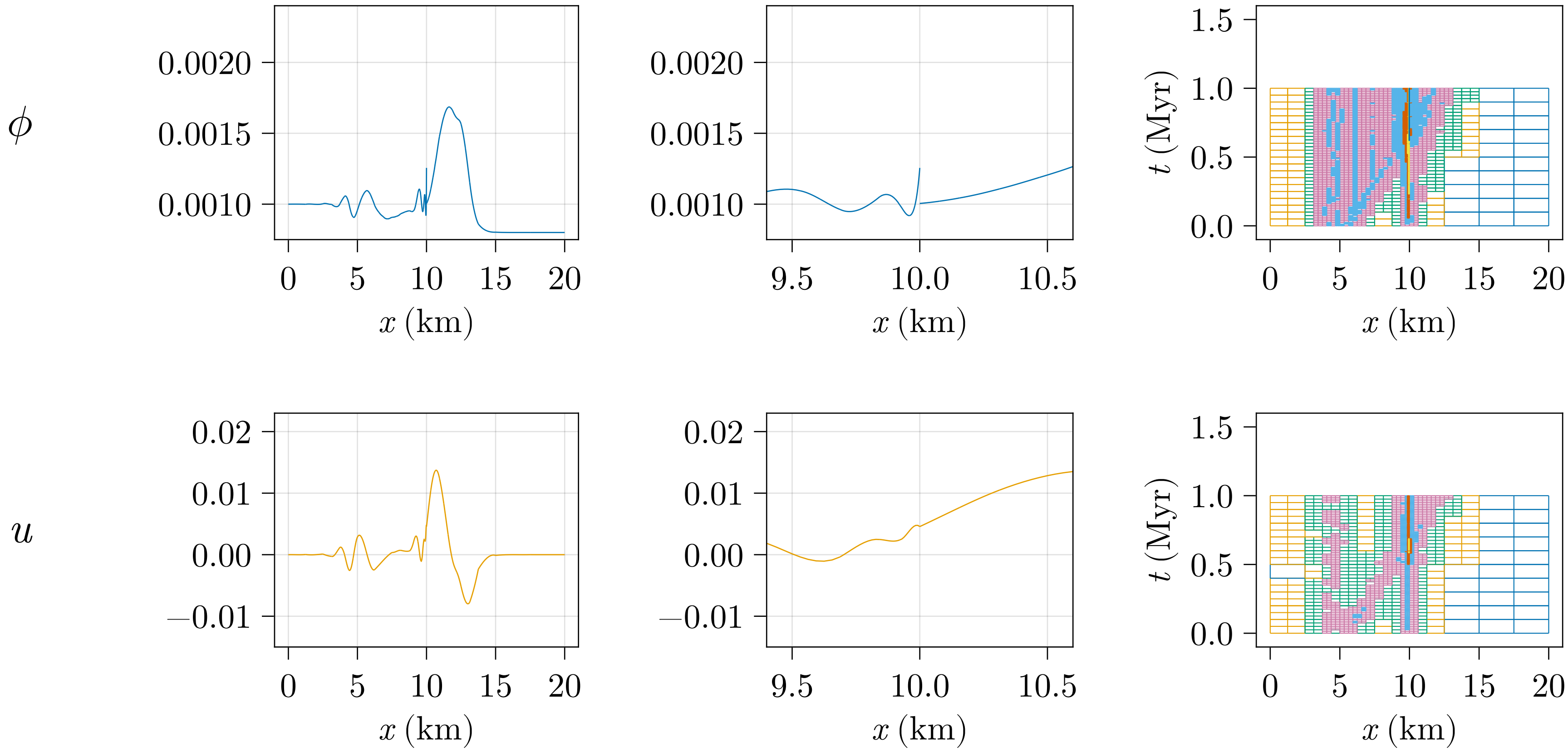

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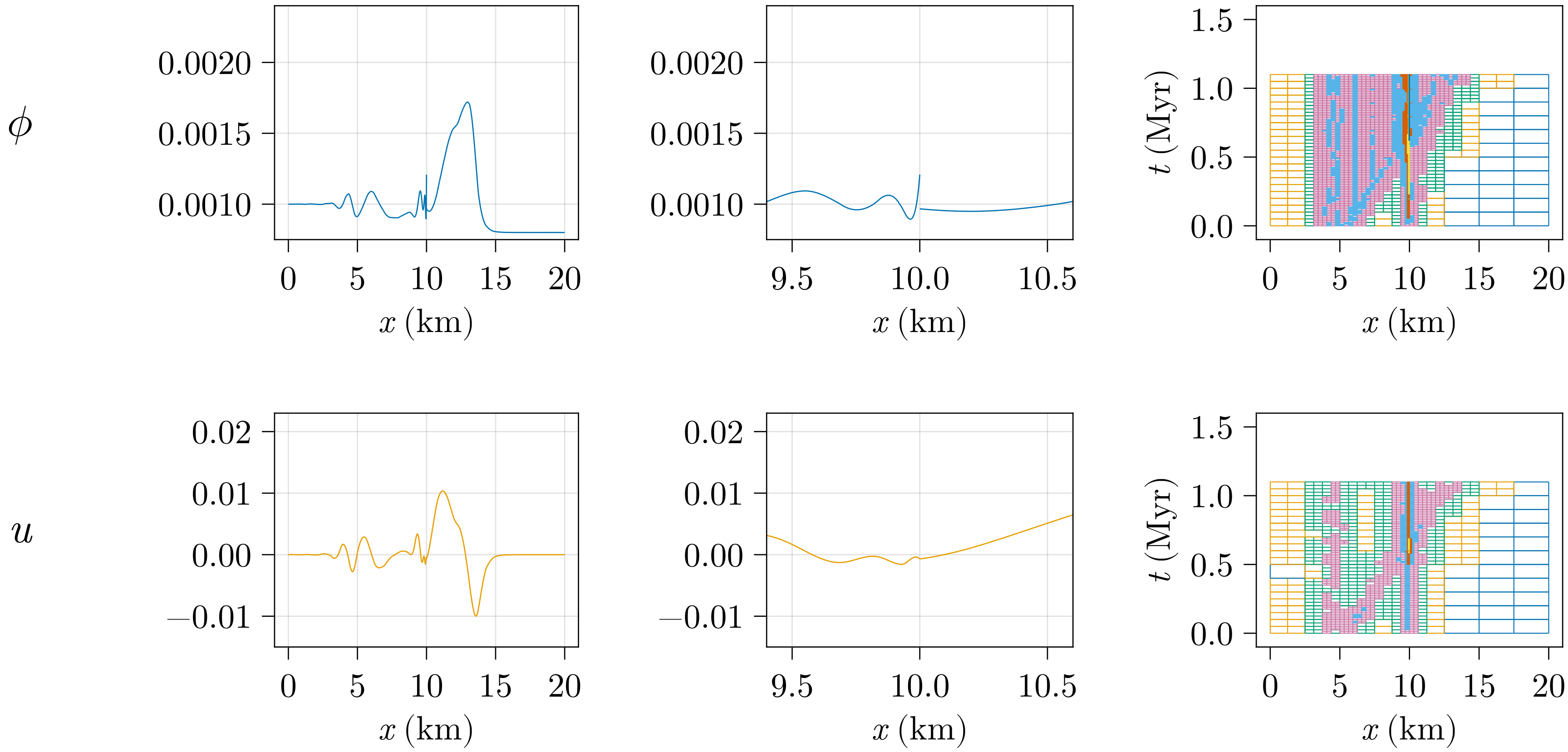

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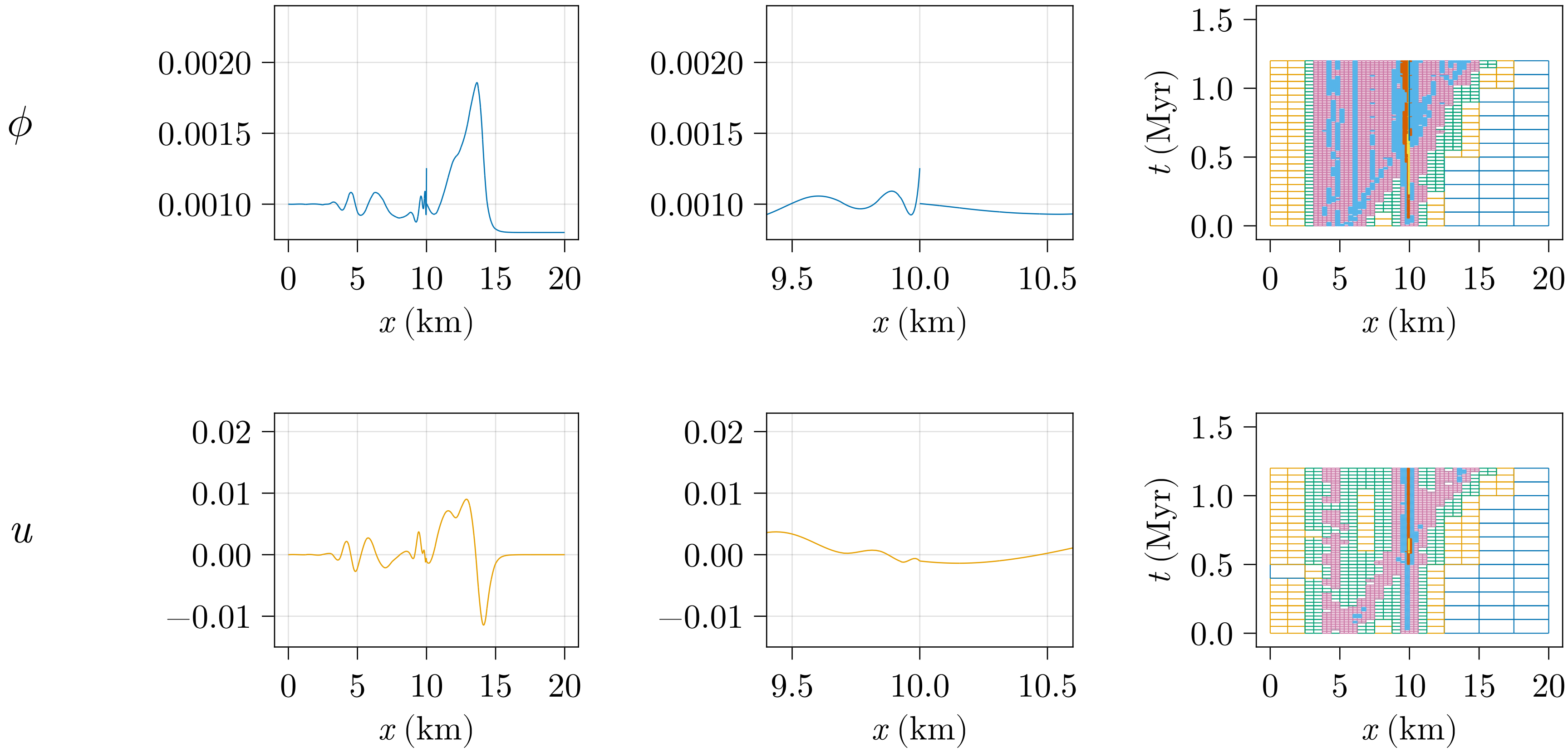

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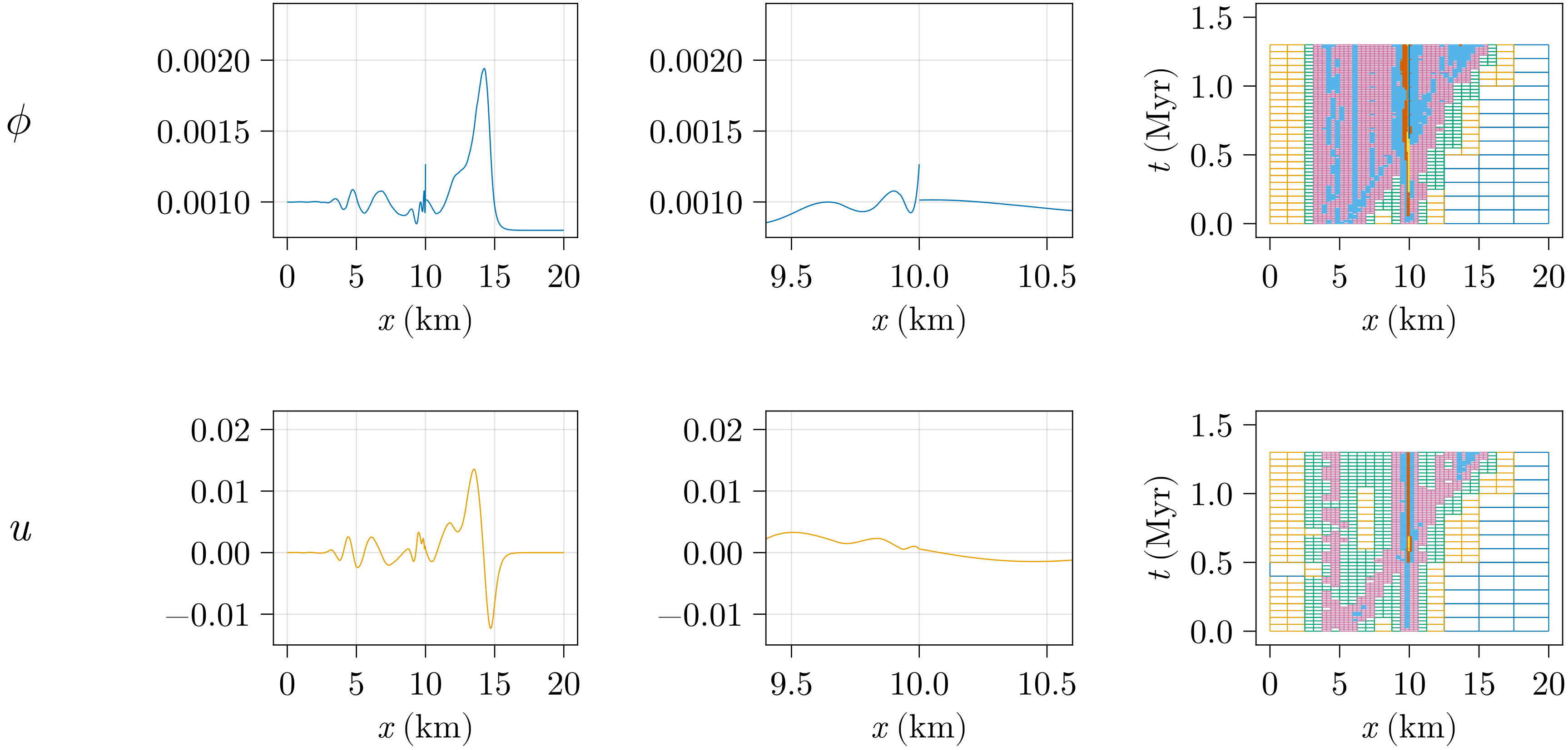

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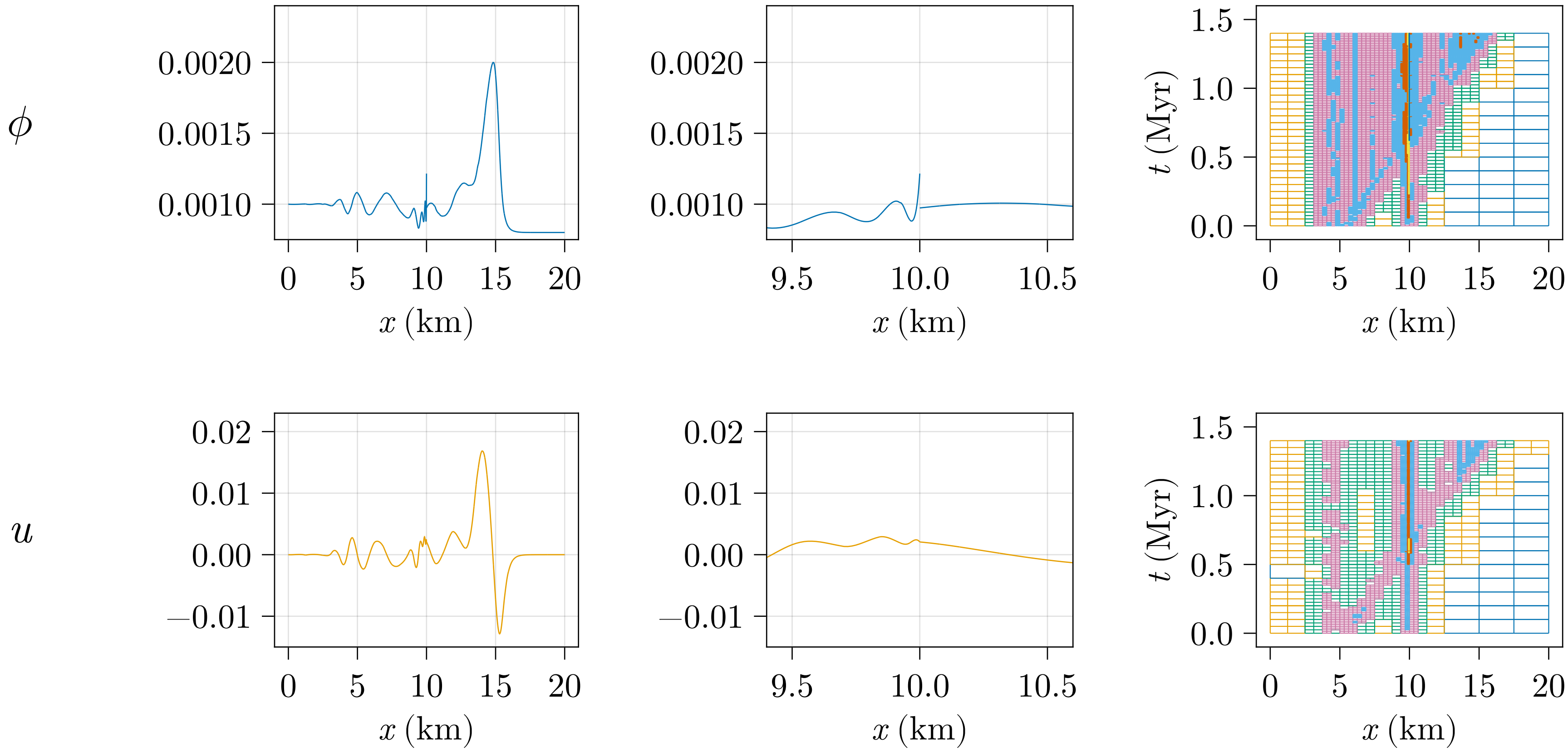

Viscous limit