Least squares rational interpolation to holomorphic functions.

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joint work with

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- Of course, Padé approximants can be defined at any point of analyticity of f in $\overline{\mathbb{C}}$ other that 0.

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- Over the last decade, Padé approximants tend to be superseded in modeling and engineering by least squares substitutes, one of which is the object of this talk.
- Let us first review basic facts on the convergence of Padé approximants.

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- But again, multipoint Padé approximants often fail to converge locally uniformly, due to spurious poles. For instance [Yattselev & LB, 2009] shows this is generic for Cauchy integrals over non-analytic arcs, whatever the interpolation scheme.
- In fact, Padé approximants are not seen best through spectacles of uniform convergence: they converge *in capacity*.

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Stronger is the notion of *geometric* convergence in capacity:

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With the previous notation, one says that Π_n converges geometrically to f in capacity on Ω iff, for each compact $K \subset \Omega$ there is $0 < \delta < 1$ such that, for every $\varepsilon > 0$, one has for n large enough:

$$|f(z) - \Pi_n(z)| < \delta^n \text{ on } K \setminus E_n \text{ with cap } E_n < \varepsilon.$$

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- When f is expressed from the outset as a Cauchy integral on such systems of cuts, analyticity assumptions can be relaxed.
- Since then Riemann-Hilbert approaches yielded refined asymptotics; we do not lean on them [Aptekarev, Yattselev,...].

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- Note that when $\Sigma = \mathbb{S}^2$ there is a sequence of rational approximants converging locally uniformly faster than geometric to f on $\Sigma \setminus E$, whose poles tend to E [Yattselev,LB]; but it is unclear how to construct such a sequence from pointwise values of f.

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- Using Rouché's theorem and properties of the logarithmic capacity, it can be shown that if f has a pole then Π_n must have a pole nearby as $n \to \infty$.
- Hence, Padé approximation is appealing to detect singularities of a meromorphic function knowing pointwise values.

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- Spurious poles are no obstacle to convergence in capacity: for if Q_n has a zero P_n will have a zero nearby, and P_n/Q_n will not be large on a big set
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- Such applications are common to Physicists and Engineers in Electromagnetism and circuit theory, but today least squares versions of Padé approximants are favored by practitioners. Let us now explain one of them.

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- For $n \in \mathbb{N}$ and $N \in \mathbb{N}$ such that $2n+1 \leq N$, we consider the criterion:

$$\begin{aligned} J_0 : \mathcal{P}_n \times \mathcal{P}_n^0 &\longrightarrow \mathbb{R}_+ \\ (p,q) &\longmapsto \sum_{i=0}^{N-1} \left| c_i(p,q,f) \right|^2 , \end{aligned} \tag{1}$$

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- A least square substitute (LSS) Padé approximant of order n on N terms to f is now $p_{n,N}^0 / q_{n,N}^0$,

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- Of course, the point 0 could be traded for any other in Ω.

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- note the monic normalization of the denominator.

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- Hereafter we present an analog of the Nutall-Pommerenke theorem for LSS Padé approximants.
- Our results require that $N = O(n)$, though from probabilistic considerations one may surmise that $N = O(n^2)$ is enough.

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- Moreover, for any f and $a > 2$ there exists a sequence (N_k, n_k) of integers, $2(a-1)n_k \leq N_k \leq 2an_k$ such that $p^0_{n,N}/q^0_{n,N}$ is unique.

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- When dealing with asymptotics of $p^0_{n,N}/q^0_{n,N}$ or $p_{n,N}/q_{n,N}$, we always assume that LSS (multipoint) Padé approximants are unique along the considered sequences (n, N).

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Theorem

Let F be a compact set in $\mathbb{C}$ such that $\operatorname{cap}(F) = 0$ and $0 \notin F$. Let f be analytic in $\overline{\mathbb{C}} \setminus F$. Then for all compact sets $K \subset \mathbb{C}$, for all $\varepsilon > 0$, $\delta > 0$, $\mu > 1$, there exists $m_0 \in \mathbb{N}$ such that for all natural numbers n and N which satisfy:

$$2n + 1 \leq N \leq \mu n,$$

$$n_0 \leq n,$$

it holds that

$$|f - p^0_{n,N}/q^0_{n,N}| < \varepsilon^m,$$

on $K \setminus E_n$ with $\operatorname{cap}(E_n) \leq \delta$.

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- Write the optimality condition for the convex minimization defining LSS Padé, with $\{c_j\}$ the Taylor coefficients of $p - qf$:

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- define two polynomials $a(z) := \sum_{k=0}^{N-2n-1} a_t z^t$ and $b(z) := \sum_{\ell=0}^{n-1} b_\ell z^\ell$ in $\mathcal{P}_{N-2n-1}$ and $\mathcal{P}_{n-1}$ respectively, with

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  Possible since $N - n - 1$ equations and $N - n$ unknowns.

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- For Γ a cycle surrounding D_η, we get for $z \notin D_\eta$ by the residue formula that

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- Consequently, by the optimality condition and Cauchy's theorem:

$$\frac{h^\ell(z)(p^0_{n,N} - q^0_{n,N} f)(z) a(z)}{z^N} = -\frac{1}{2i\pi}\int_\Gamma \frac{h^\ell(\xi) q^0_{n,N}(\xi) f(\xi) a(\xi)}{\xi^N(\xi - z)} d\xi.$$

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From there we get the estimate:

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- along with the fact that $|\xi| > c > 0$.
- We also need that for N large enough, a cannot be zero unless f is rational.

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- Maybe for other types of convergence? Probabilistic ones?

And most importantly

THANK YOU.