Stable High-Order Approximation of Triangulated Manifolds for Surface Integration

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Abstract

We propose a novel method for deriving High-Order Volume Elements (HOVE) for scalar function integration on regular embedded manifolds. Using square-squeezing, a transformation that reparametrizes a flat triangulation mesh into a quad mesh, we approximate the integrand and volume element within each hypercube via Chebyshev–Lobatto grid interpolation.

Contribution

- **Quadrilateral re-parametrization:** For a flat triangulation $\mathcal{T}_h$ of Γ, each simplex is reparametrized via a hypercube-to-simplex map $\sigma : [-1,1]^d \to \triangle_d$, termed *square-squeezing*. The ϱ_i are interpolated on each hypercube using k^{th}-order tensorial Chebyshev–Lobatto nodes.
- **Error Bound Estimation**: Theoretical error estimates showing $\mathcal{O}(n^{-r}) + \mathcal{O}(k^{-(r-1)})$ for smooth surfaces, ensuring exponential convergence rates.
- **Computational Efficiency**: Application of FFT-based differentiation and interpolation for $\mathcal{O}(N \log N)$ operations, significantly improving numerical stability for complex surfaces.

Introduction

We consider a compact, orientable, d-dimensional C^{r+1}-manifold Γ embedded in an m-dimensional Euclidean space ($0 \le d \le m$), and an integrable function $f : \Gamma \to \mathbb{R}$. This work introduces a new algorithm for approximating the surface integral:

$$\int_\Gamma f(\mathbf{x})\, dS. \tag{1}$$

Assuming that the smooth surface Γ is topologically equivalent to a d-dimensional polyhedral surface Γ_h, composed of simplices $\mathcal{T}_h = \{T_i\}, i = 1, \dots, K$:

$$\Gamma_h = \bigcup_{T_i \in \mathcal{T}_h} T_i.$$

The maps $\varrho_i = \pi_i \circ \tau_i : \triangle_d \to \mathbb{R}^{d+1}$ define the partition of Γ. A key challenge is the distribution of nodes within simplices for stable high-order polynomial interpolation, as no direct analogue of the Chebyshev-Lobatto rule exists for triangles. To address this, we use a hypercube-to-simplex transformation $\sigma : [-1,1]^d \to \triangle_d$, known as *square-squeezing* (Eq. (2)).

Tensorial interpolation

Let $n \in \mathbb{N}$, and define the tensorial Chebyshev–Lobatto grid $G_{d,n} = \bigoplus_{i=1}^d \mathrm{Cheb}_n$, where

$$\mathrm{Cheb}_n = \left\{ \cos\left(\frac{k\pi}{n}\right) : 0 \le k \le n \right\},$$

indexed by $A_{d,n} = \{\alpha \in \mathbb{N}^d : \|\alpha\|_\infty \le n\}$. For each $\alpha \in A_{d,n}$, the tensorial multivariate Lagrange polynomials are

$$L_\alpha(x) = \prod_{i=1}^{d} l_{\alpha_i,i}(x), \quad l_{j,i}(x) = \prod_{k=0,k\neq j}^{n} \frac{x_i - p_{k,i}}{p_{j,i} - p_{k,i}}.$$

Given a function $f : [-1,1]^d \to \mathbb{R}$, the interpolant $Q_{G_{d,n}} f \in \Pi_{d,n}$ of f in $G_{d,n}$ is

$$Q_{G_{d,n}} f = \sum_{\alpha \in A_{d,n}} f(p_\alpha) L_\alpha.$$

Quadrilateral re-parametrization

Given a triangulation $\mathcal{T}_h$ of the surface $\Gamma = Q_\Gamma^{-1}(0)$, for each triangle $T_i \in \mathcal{T}_h$, $i = 1, \dots, K$, we consider a quad re-parametrization

$$\varphi_i : [-1,1]^2 \to T_i, \quad \varphi_i = \varrho_i \circ \sigma = \pi_i \circ \tau_i \circ \sigma, \quad i = 1, \dots, K,$$

The quadrilateral re-parametrization enables interpolating the geometry functions $\varphi_i = \varrho_i \circ \sigma : [-1,1]^2 \to \mathbb{R}^3$ by tensor-product polynomials. We derive the k^{th}-order polynomial interpolant $Q_{\varphi_i,k}$ of φ_i in tensor-product polynomials Chebyshev-Lobatto nodes.

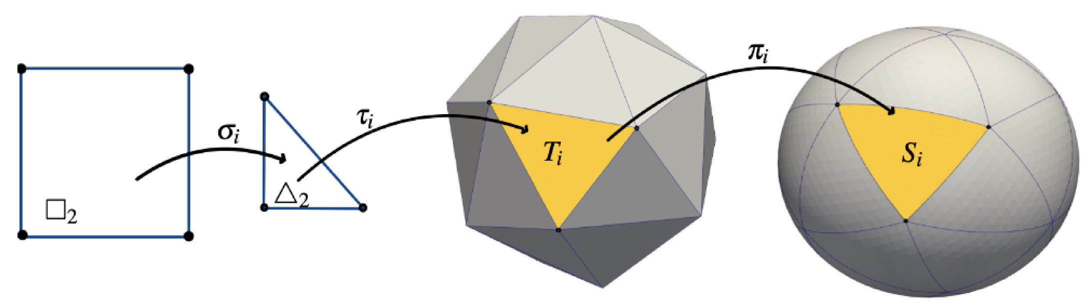


Figure 1: Construction of a surface parametrization over $\triangle_2$ by closest-point projection from a piecewise affine approximate mesh, and re-parametrization over the square $\square_2 := [-1,1]^2$.

Consequently, the integral is approximated by numerically computing

$$\sum_{i=1}^{K} \int_{[-1,1]^d} Q_{G_{d,n}}(f \circ \varphi_i)(\mathbf{x}) \sqrt{\det\left((DQ_{G_{d,k}}\varphi_i(\mathbf{x}))^T DQ_{G_{d,k}}\varphi_i(\mathbf{x})\right)}\, d\mathbf{x}$$

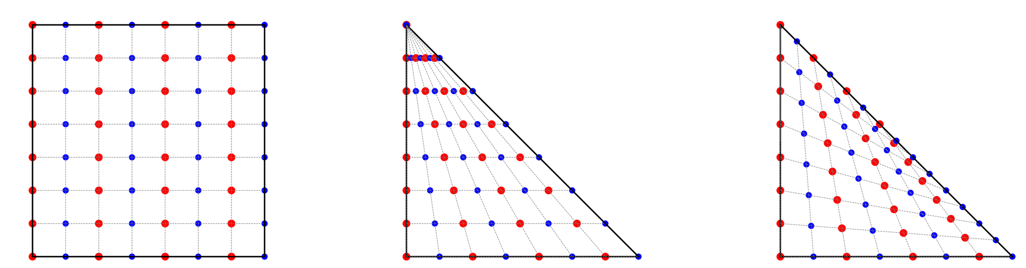
Square-squeezing map

We re-scale $[-1,1]^2$ to $[0,1]^2$ by setting $\tilde{x}_1 = (x_1+1)/2$, $\tilde{x}_2 = (x_2+1)/2$. The square-squeezing transformation on $[0,1]^2$ becomes

$$\sigma : [0,1]^2 \to \triangle_2, \quad \sigma(\tilde{x}_1, \tilde{x}_2) = \left(\tilde{x}_1 - \frac{\tilde{x}_1\tilde{x}_2}{2},\ \tilde{x}_2 - \frac{\tilde{x}_1\tilde{x}_2}{2}\right)^T. \tag{2}$$

Duffy transformation

$$\sigma_{\text{Duffy}} : [-1,1]^2 \to \triangle_2, \quad \sigma_{\text{Duffy}}(x,y) = \left(\frac{1}{4}(1+x)(1-y), \frac{1+y}{2}\right), \tag{3}$$



(a) Standard square (b) Duffy (c) Square-Squeezing

Figure 2: Bilinear square–simplex transformations: Deformations of equidistant grids, under Duffy's transformation (b) and square-squeezing (c)

where $Q_{G_{d,k}}\varphi_i(\mathbf{x})$ denoting a $k-$th order polynomial approximating the map φ_i, whereas $Q_{G_{d,n}}(f \circ \varphi_i)(\mathbf{x})$ is a $n-$th order polynomial approximating the integrand $f : \Gamma \to \mathbb{R}$.

Theoretical Error Estimates

Theorem

Let Γ be a smooth surface and $f : \Gamma \longrightarrow \mathbb{R}$ be a integrable function. Consider a piecewise linear triangulation $\mathcal{T}_h$ of Γ with maximum triangle diameter $h > 0$. For each triangle $T_i \in \mathcal{T}_h$, $i = 1, \dots, K$ denote with $Q_{G_{d,k}}\varphi_i$ the k^{th}-order approximation of φ_i. Then

$$\left| \int_\Gamma f\, dS - \sum_{i=1}^{K} \int_{[-1,1]^d} Q_{G_{d,n}}(f \circ \varphi_i)(\mathbf{x}) \sqrt{\det\left((DQ_{G_{d,k}}\varphi_i(\mathbf{x}))^T DQ_{G_{d,k}}\varphi_i(\mathbf{x})\right)}\, d\mathbf{x} \right| = \mathcal{O}\left(n^{-r}\right) + \mathcal{O}\left(k^{-(r-1)}\right),$$

where $Q_{G_{d,n}}(f \circ \varphi_i)(\mathbf{x})$ is a $n-$th order polynomial approximating the integrand $f : \Gamma \to \mathbb{R}$.

Results

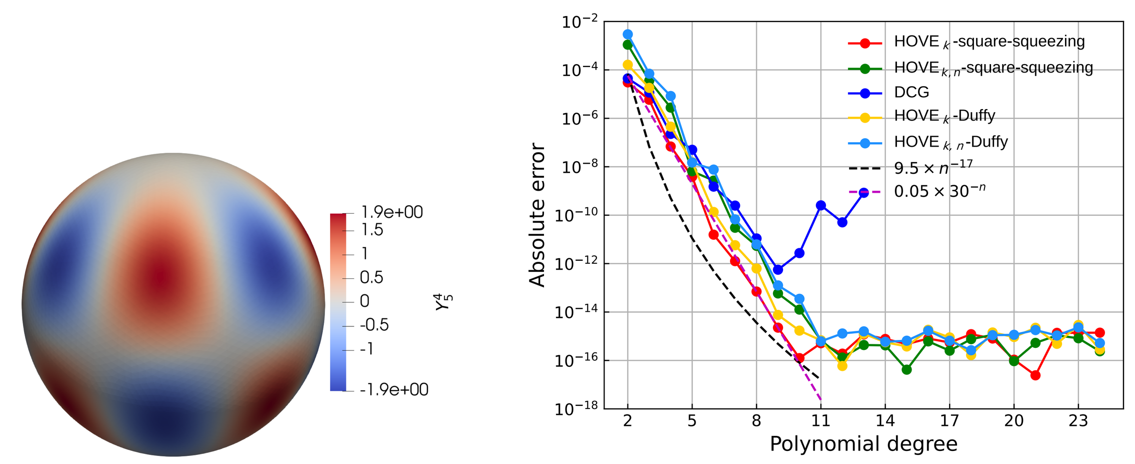


Figure 3: Visualization of the spherical harmonic Y_5^4 (left). Integration errors of DCG and *HOVE* with respect to the interpolation degree. Abbreviations: *HOVE$_k$* – interpolating only the geometry, *HOVE$_{k,n}$* – interpolating the geometry and the integrand.

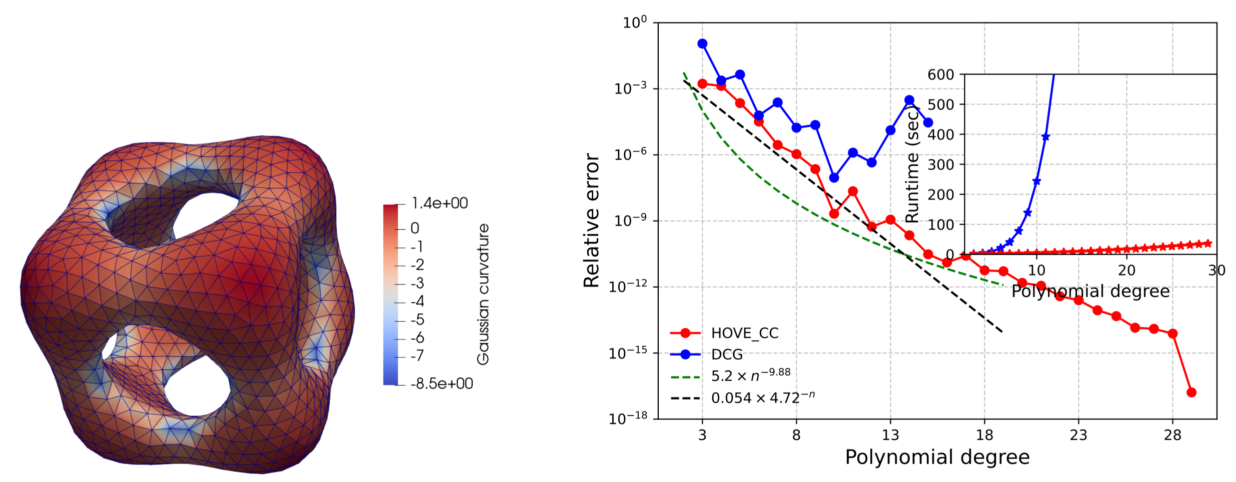


Figure 4: Gauss-Bonnet validation $\int_\Gamma K_{\text{Gauss}}\, dS = 2\pi\chi(\Gamma)$ for the Swiss cheese block composed of 2944 triangles.

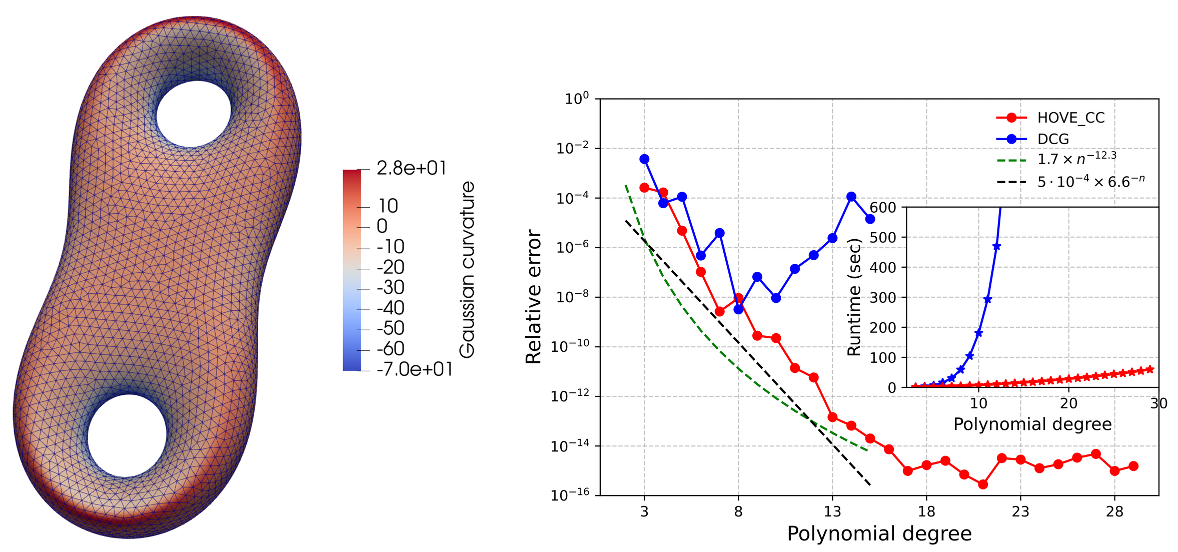


Figure 5: Gauss-Bonnet validation for a double torus composed of 8360 triangles.

References

[1] Gentian Zavalani and Michael Hecht.
High-order numerical integration on regular embedded surfaces, 2024.

[2] Gentian Zavalani, Oliver Sander, and Michael Hecht.
High-order integration on regular triangulated manifolds reaches super-algebraic approximation rates through cubical re-parameterizations, 2023.

[3] Simon Praetorius and Florian Stenger.
Dune-curvedgrid – a dune module for surface parametrization.