"de Casteljau algorithm" and "Bézier curves": When ? Who ? How ?

Christophe Rabut, INSA and IMT, Toulouse, France

Context: 1. Numerically controled tooling arrives, and so industry needs transform designers' drawings into points sequences.
2. April 1958 Citroën hires a young mathematician. 1960 (or 1959?) Renault entrusts the tooling service head to do the job

Result: The true "invention" (or start) of "CAO", "CFAO". Two completely different approaches for a same result...
Quite at the same time, without knowing each other, both in a car industry... at a few kilometers distance !

Paul de Faget de Casteljau ; "Enveloppe Soleau"

Born 1930 (died 2022)

Brilliant student at "Ecole Normale Supérieure", Paris

Knows nothing on car tooling and manufacturing

April 1st, 1958: hired by Citroën (first job)

December 1958: Citroën internal report, followed by

February 1959: "Enveloppe Soleau" (ES) on "Possible Improvements to the Techniques of Rating and of Numerical Computation"

Citroën imposed secret on this:
ES is a sealed envelope for proving as certain the date of an invention, idea or creation of a work. It is applied to and kept by the French *"Institut National de la Propriété Industrielle"*

Soleau Envelope: Motivation and first algorithm

Motivation and key points

Easy computation with some approximation may be better than precise results long to compute.

"Simple functions" $\Longrightarrow$ parametric polynomials... Try interpolation?

A triangular algorithm for (degree 3) polynomial p extrapolation

$q(x) = p(x+\frac{h}{2}) - p(x-\frac{h}{2}); r(x) = q(x+\frac{h}{2}) - q(x-\frac{h}{2}); s(x) = r(x+\frac{h}{2}) - r(x-\frac{h}{2})$

Then q and r are degree 2 and 1, s is a constant.

Let $P_i = p(x_0 + i\,h)$; $Q_{i/2} = q(x_0 + i\,h/2)$); $R_i = r(x_0 + i\,h)$; $S_{i/2} = s(\ldots)$

The following algorithm computes $P_4, P_5, \ldots$

P_0
$Q_{1/2} = P_1 - P_0$
P_1 $\quad R_1 = Q_{3/2} - Q_{1/2}$
$Q_{3/2} = P_2 - P_1$ $\quad S_{3/2} = R_2 - R_1 = K$
P_2 $\quad R_2 = Q_{5/2} - Q_{3/2}$
$Q_{5/2} = P_3 - P_2$ $\quad S_{5/2}$
P_3 $\quad R_3 = K + R_2$
$Q_{7/2} = R_3 + Q_{5/2}$
$P_4 = P_3 + Q_{7/2}$

Simple generalization to any degree

"Barycentric method" ("de Casteljau' algorithm")

Two problems of the first algorithm:

1. This is extrapolation, not interpolation
2. They are "stupid" oscillations $\Longrightarrow$ out of the sheet of paper!

How to modify the algorithm to stay inside the sheet of paper?

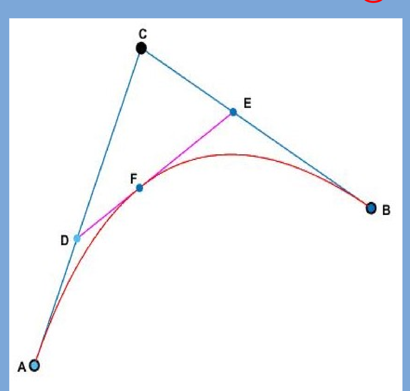
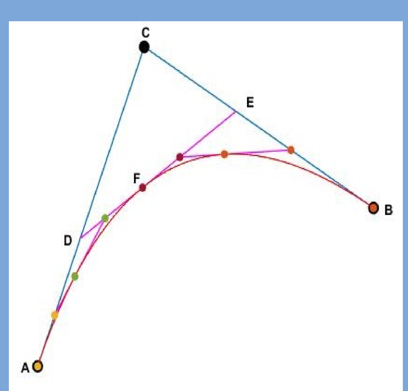
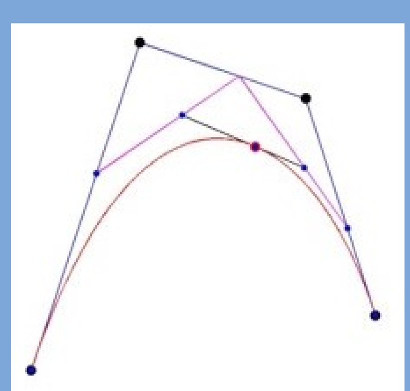
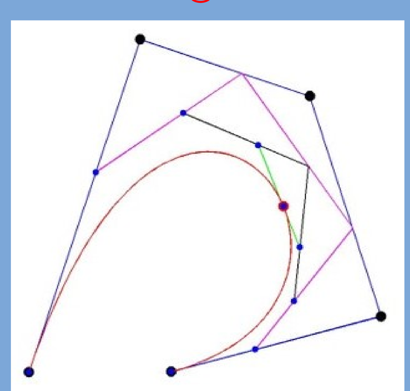
Idea: Change the differences into a sum!... and
Choose a point in the segment linking the two generating points
ie $Q_{1/2}$ inside P_0 and P_1 as for example $Q_{1/2} = (P_0 + P_1)/2$
Even better : somewhere in the interval $[P_0..P_1]$,
which is $Q_{1/2} = \alpha P_0 + \beta P_1$ with $\alpha + \beta = 1$ **"Barycenter"**

P_0
$Q_{1/2} = \alpha\, P_1 + \beta\, P_0$
P_1 $\quad R_1 = \alpha\, Q_{3/2} + \beta\, Q_{1/2}$
$Q_{3/2} = \alpha\, P_2 + \beta\, P_1$ $\quad S_{3/2} = \alpha\, R_2 + \beta\, R_1$
P_2 $\quad R_2 = \alpha\, Q_{5/2} + \beta\, Q_{3/2}$
$Q_{5/2} = \alpha\, P_3 + \beta\, P_2$
P_3

He looses interpolation... but "It is not so important"
Simple generalization to any degree

Figures and Consequences

1. Last point equation: $S_{3/2} = \sum_{i=0:n} \binom{n}{i} \alpha^i \beta^{n-i} P_i$
$\alpha = x;\ \beta = 1 - x \Rightarrow p(x) = \sum_{i=0:n} \binom{n}{i} x^i (1-x)^{n-i} P_i$
2. Iteration possible with $\{P_0\,,\ Q_{1/2}\,,\ R_1\,,\ S_{3/2}\}$ and $\{S_{3/2}\,,\ R_2\,,\ Q_{5/2}\,,\ P_3\}$
3. This is a nice generalization of the construction of a parabola from two points and their tangents.
4. Beautiful 3D generalization with barycentric coordinates in a triangle

$\alpha = .6, \beta = .4$ Parabola degree 3 degree 4

Pierre Bézier: main ideas

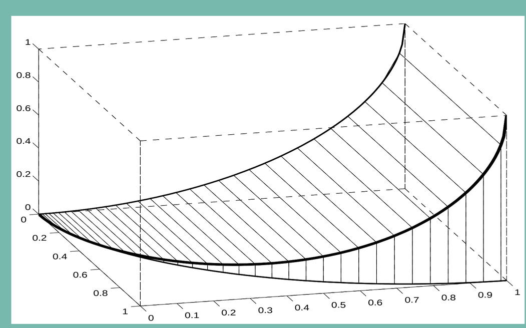
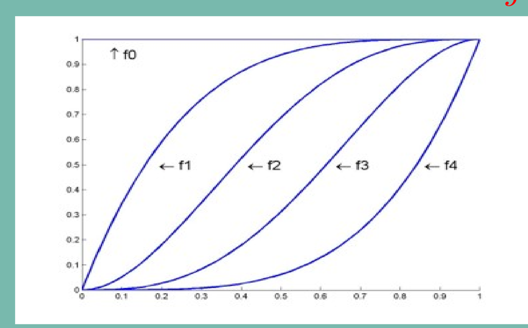
Born 1910 (died 1999)
Student at ENSAM and Ecole Sup'Elec (Engineering schools)
1933: Hired by Renault... He later became the head of the tool department.

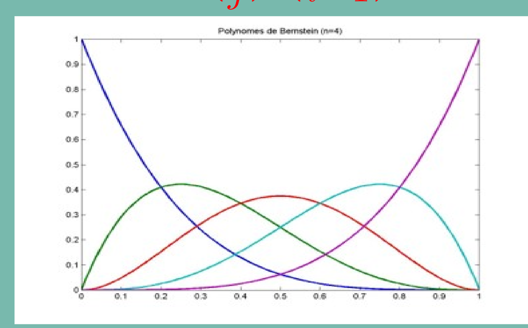
He introduced three structural improvements:
Automatic machines controled by sequential electromagnetic relays.
"Transfer lines": specialized machines able to execute a big number of machining operations of diverse nature in an entirely automated way.
1958: the first numerically controled drill

1960: Main ideas to solve the problem posed by Renault:
1. "Erase the blackboard and restart from scrach"
Which means: do not try to copy or imitate an existing curve, but create a curve directly numerically
2. Use parametric polynomials,
3. Modify the shape of a curve by modifying the coordinate axes.
4. Put vectors end to end instead of all starting at origin

"Onésime Durand's functions" (around 1961-1962 (?))

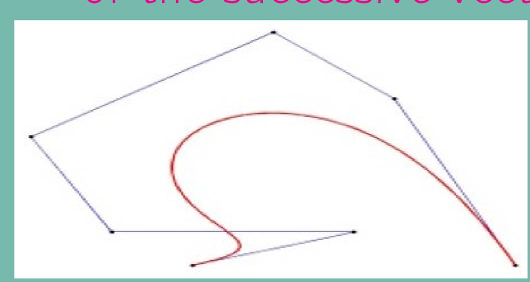
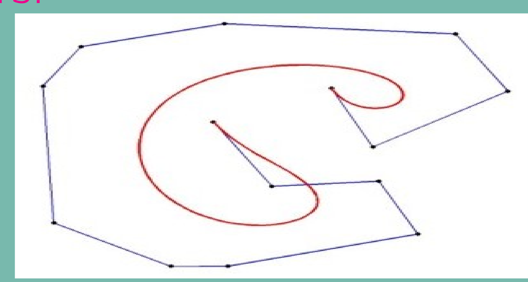
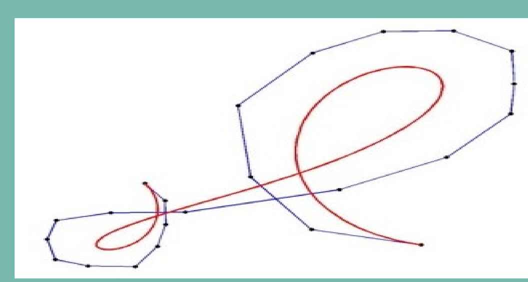
We need a "good basis" for curves...
Idea : Intersection of two cylinders... and a specific polynomial to imitate it: "O. Durand functions" $f_i^n(t) = \sum_{j=i}^n (-1)^{i+j} \binom{n}{j} \binom{j-1}{i-1} t^j$



Two cylinders intersect. O.D. $f^n s$ $(n=4)$ Bernstein $f^n s$ $(n=5)$

Then $F(t) = P_0 + \sum_{i=1}^n \overrightarrow{P_{i-1}P_i}\ f_i^n(t)$ should follow the shape of the successive vectors.

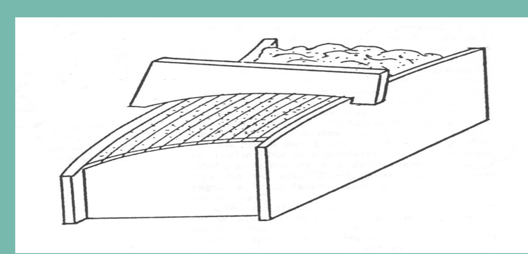




Bernstein polynomials; surfaces

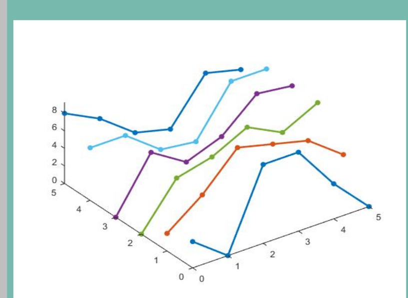
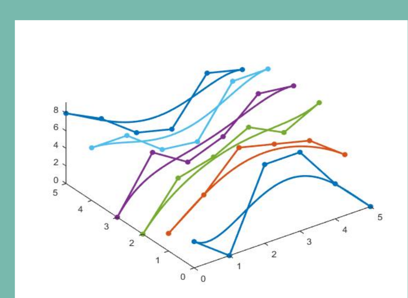
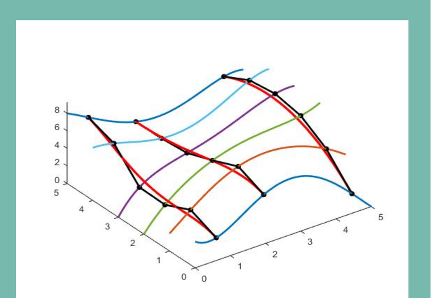
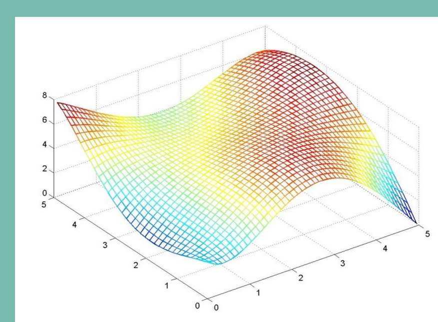
As later ($\simeq$ 1970) pointed out by Claude Riaux (a Pierre Bézier's colleague), the points P_i occur twice: once in $\overrightarrow{P_{i-1}P_i}$ and once in $\overrightarrow{P_iP_{i+1}}$.
So $F(t) = \sum_{i=0}^n P_i\ (f_i^n(t) - f_{i+1}^n(t)) = \sum_{i=1}^n P_i\, B_i^n(t)$
where $B_i^n(t) = \binom{n}{i} t^i\,(1-t)^{n-i}$ is the i^{th} degree n Bernstein polynomial.

Surfaces

1. **For rectangles:** "tensorial product":



$S(u,v) = \sum_{i=0}^m \sum_{j=0}^n P_{ij}\, B_i^m(u)\, B_j^n(v)$

2. For triangles: limit of a rectangle when a vertex tends to a neighbour vertex

In short : Pierre and Paul

Paul de Casteljau, a mathematician who also did pragmatic engineering and who continued developing interesting new mathematics.
Pierre Bézier, a pragmatic engineer who needed mathematics, became a mathematician, and developed his method ("UNISURF") for car manufacturing.

Both independently developed a mathematical method to derive curves and surfaces fitting any kind of shape. PdC also developed a very efficient algorithm to derive these curves, which ensured their success.

Both their names entered in the history of mathematics, of CAGD and of their applications. PdC was the first (curves and algorithm), but due to the secret imposed by Citroën, the curves are named after PB, the algorithm after PdC.

A prophetic sentence written by PdC in the first part of the "Enveloppe Soleau" (1959): *"We will see in the following a very interesting application of the notion of barycentre, which undoubtedly makes it possible to mathematically fix forms which were previously governed by aesthetics."*