Pseudo-Reversing and Its Application to Manifold-Valued Multiscaling

Wael Mattar and Nir Sharon
waelmattar@tauex.tau.ac.il

Abstract

The well-known Wiener's lemma is a deep and valuable statement in harmonic analysis; in the space of functions with absolutely convergent Fourier series, elements that admit a multiplicative inverse are called reversible. We present a method called **pseudo-reversing** for approximating the reverse of functions that are not necessarily reversible.

Next, we make use of pseudo-reversing and define downsampling operators that enable us to construct a multiscale pyramid transform, which is a tool for representing data on different scales in a hierarchical fashion. Finally, we demonstrate the application of **contrast enhancement** via multiscaling to manifold-valued data.

Wiener's lemma

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle of the complex plane, and denote by $\mathcal{A}(\mathbb{T})$ the Banach space consisting of all periodic functions $f(t) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi ikt}$ with coefficients $\boldsymbol{a} \in \ell_1(\mathbb{Z})$. We endow $\mathcal{A}(\mathbb{T})$ with the norm

$$\|f\|_{\mathcal{A}} = \|\boldsymbol{a}\|_1 = \sum_{k\in\mathbb{Z}} |a_k|.$$

Wiener's lemma. If $f \in \mathcal{A}(\mathbb{T})$ and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then also $1/f \in \mathcal{A}(\mathbb{T})$. That is, $1/f(t) = \sum_{k\in\mathbb{Z}} b_k e^{2\pi ikt}$ for some $\boldsymbol{b} \in \ell_1(\mathbb{Z})$.

Pseudo-reversing and polynomials

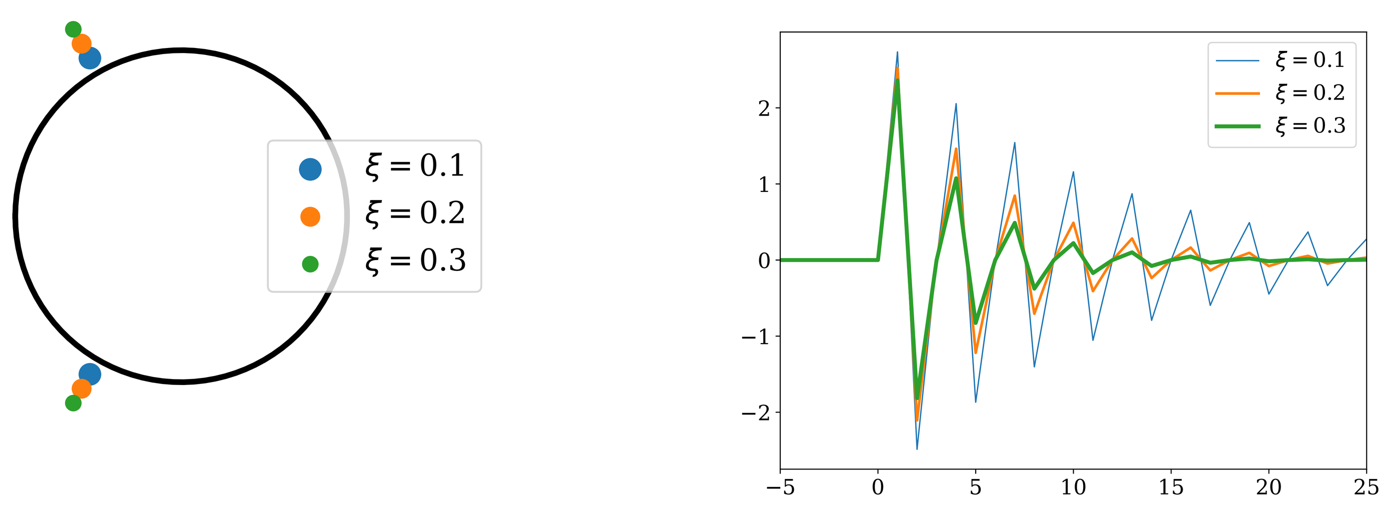
Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \in \mathbb{N}$. We assume that the coefficients of p sum to 1 and rewrite p as

$$p(z) = C(p) \prod_{r\in\Lambda} (z - r),$$

where Λ the set of all zeros of p including multiplicities. For some $\xi > 0$, the **pseudo-reverse** of the polynomial p is defined by

$$p_\xi^\dagger(z) = \left(C(p_\xi^\dagger) \prod_{r\in\Lambda\setminus\mathbb{T}} (z - r) \prod_{r\in\Lambda\cap\mathbb{T}} (z - (1+\xi)r) \right)^{-1},$$

where $C(p_\xi^\dagger)$ is a constant depending on ξ determined by $p_\xi^\dagger(1) = 1$.



(a) Displacements of zeros.

(b) Pseudo-reverse coefficients.

Proposition 1. For any polynomial p, the product $p_\xi^\dagger p$ converges in norm to 1 as $\xi \to 0^+$.

$$\lim_{\xi\to0^+} \|p_\xi^\dagger p - 1\|_{\mathcal{A}} = 0.$$

Proposition 2. If all zeros of the polynomial p are on the unit circle, then $p_\xi^\dagger(z)$ converges uniformly to 1 as $\xi \to \infty$ on every compact subset of $\mathbb{C}$.

The reversibility condition $\kappa : \mathcal{A}(\mathbb{T}) \to [1, \infty]$ acting on a function $f \in \mathcal{A}(\mathbb{T})$ is defined by

$$\kappa(f) = \frac{\sup_{z\in\mathbb{T}} |f(z)|}{\inf_{z\in\mathbb{T}} |f(z)|},$$

with the convention $\kappa(f) = \infty$ for functions with $\inf_{z\in\mathbb{T}} |f(z)| = 0$. It is evidently seen in [1] that if f is reversible, positive and band limited, then the coefficients $\boldsymbol{b}$ of $1/f$ obey

$$|b_k| \leq C\lambda^{|k|}, \quad k \in \mathbb{Z},$$

for some $C > 0$ and $0 < \lambda < 1$ depending on $\kappa(f)$.

Corollary. Let p be a polynomial with $n \in \mathbb{N}$ zeros all on the unit circle. Then

$$\kappa(p_\xi^{-\dagger}) \leq (1 + 2/\xi)^n$$

for any $\xi > 0$.

Pseudo-reversing refinement operators

Given a subdivision scheme $\mathcal{S}_{\boldsymbol{\alpha}}$ with mask $\boldsymbol{\alpha}$, its reverse *decimation* operator is defined via $\mathcal{D}_{\boldsymbol{\gamma}}\boldsymbol{c} = \boldsymbol{\gamma} * (\boldsymbol{c} \downarrow 2)$ for any sequence $\boldsymbol{c}$, where $\boldsymbol{\gamma}$ under the z-transform solves

$$\boldsymbol{\alpha}(z)\boldsymbol{\gamma}(z) = 1, \quad z \in \mathbb{C}.$$

We hence look for a solution $\boldsymbol{\gamma} \in \ell_1(\mathbb{Z})$.

To solve for $\boldsymbol{\gamma}$ we rely on Wiener's lemma; if $\boldsymbol{\alpha}(z)$ has zeros on $\mathbb{T}$ then we use pseudo-reversing to get an approximation $\boldsymbol{\gamma}_\xi^\dagger$ of $\boldsymbol{\alpha}^{-1}$.

Multiscaling manifold values

Let $\mathcal{M}$ be a Riemannian manifold and denote by $\boldsymbol{c}^{(k)} \subset \mathcal{M}$ sequences with indices associated with the grid $2^{-k}\mathbb{Z}$. A multiscale transform of a sequence $\boldsymbol{c}^{(J)}$ yields a pyramid representation comprises a coarse approximation $\boldsymbol{c}^{(0)}$ in addition to detail coefficients $\boldsymbol{d}^{(\ell)}$, $\ell = 1, \dots, J$.

Sequence $\boldsymbol{c}^{(J)}$ $\underset{\text{reconstruction}}{\overset{\text{decomposition}}{\rightleftarrows}}$ Pyramid $\left\{\boldsymbol{c}^{(0)}; \boldsymbol{d}^{(1)}, \boldsymbol{d}^{(2)}, \dots, \boldsymbol{d}^{(J)}\right\}$

The analysis and synthesis are done with a refinement operator $\mathcal{S}_{\boldsymbol{\alpha}}$ and its reverse decimation $\mathcal{D}_{\boldsymbol{\gamma}}$. In particular, the decomposition of a sequence $\boldsymbol{c}^{(J)}$ is defined iteratively via

$$\boldsymbol{c}^{(\ell-1)} = \mathcal{D}_{\boldsymbol{\gamma}}\boldsymbol{c}^{(\ell)}, \quad \boldsymbol{d}^{(\ell)} = \boldsymbol{c}^{(\ell)} \ominus \mathcal{S}_{\boldsymbol{\alpha}}\boldsymbol{c}^{(\ell-1)}, \quad \ell = 1, \dots, J,$$

while the inverse transform is defined via

$$\boldsymbol{c}^{(\ell)} = \mathcal{S}_{\boldsymbol{\alpha}}\boldsymbol{c}^{(\ell-1)} \oplus \boldsymbol{d}^{(\ell)}, \quad \ell = 1, \dots, J.$$

The operations above denote the exponential mapping and its inverse associated to a point $p \in \mathcal{M}$,

$$\exp_p(v) = p \oplus v \quad \text{and} \quad \log_p(q) = q \ominus p.$$

Theoretical results

- It was shown in [2] that if $\boldsymbol{c}^{(J)}$ is sampled over an arc-length parametrization grid of a differentiable curve in $\mathcal{M}$, then the detail coefficients generated by the multiscale decay geometrically.
- In case $\mathcal{S}_{\boldsymbol{\alpha}}$ is not reversible and we apply a pseudo-reverse decimation while multiscaling, the detail coefficients decay still holds but with a controllable violation depending on ξ.
- Under certain mild assumptions, the inverse multiscale transform becomes stable.

Contrast enhancement application

The manifold of interest is $\mathcal{M} = \mathrm{SO}(3)$. The subdivision scheme $\mathcal{S}_{\boldsymbol{\alpha}}$ used in multiscaling has the mask $\boldsymbol{\alpha} = 1/12\,[3, 4, 3, 4, 3, 4, 3]$. The even mask is not reversible, we used the pseudo-reverse decimation operator $\mathcal{D}_{\boldsymbol{\gamma}}$ with $\xi = 0.64$.

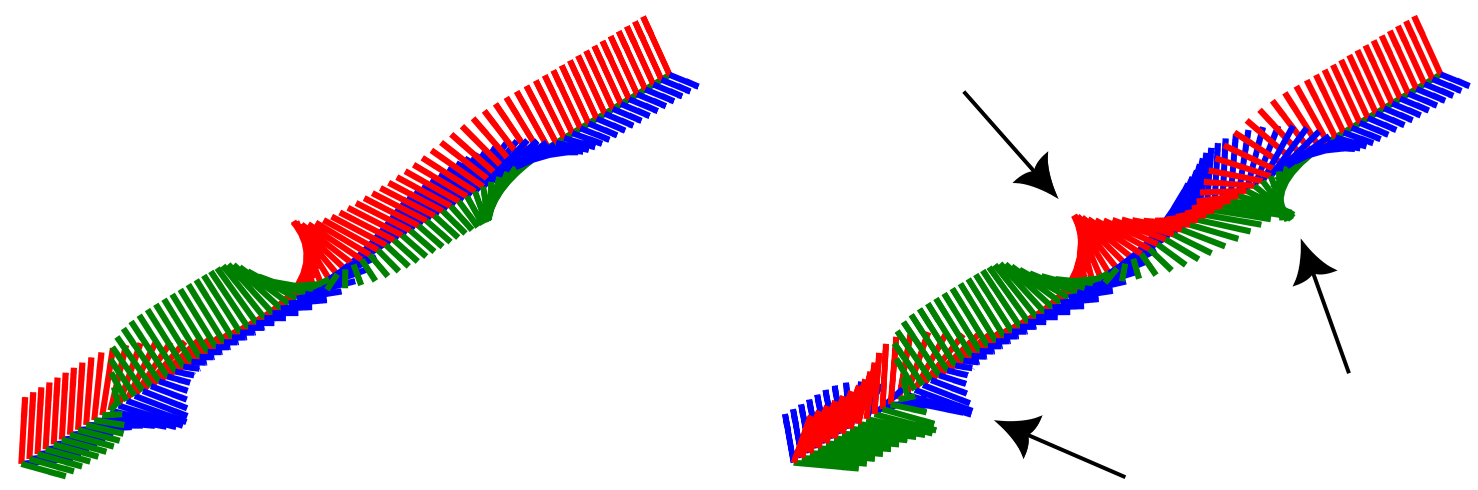


Figure 2: Contrast enhancement of SO(3)-valued sequence. On the left, the original sequence of rotation matrices. On the right, the enhanced rotation sequence. The largest 20% of the detail coefficients of each layer were enlarged by 40%. The black arrows indicate the regions with the most drastic twists – highlighting the effect of the application.

References

[1] T. Strohmer, "Four short stories about Toeplitz matrix calculations," *Linear Algebra and its Applications*, vol. 343, pp. 321–344, 2002.

[2] W. Mattar and N. Sharon, "Pyramid transform of manifold data via subdivision operators," *IMA Journal of Numerical Analysis*, vol. 43, no. 1, pp. 387–413, 2023.

The poster is based on **arxiv.org/abs/2305.06261**